Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4051-WE01

Title:
General Relativity IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. A two-dimensional space has coordinates $x^\mu = (t, x)$. New local coordinates $\tilde{x}^\mu = (u, v)$ are defined by $u = t^2 - x^2$ and $v = 2tx$.

   (a) A vector field V has components $V^\mu = (t, x)$ with respect to the old coordinates x^μ. What are its components $\widetilde{V}^\mu$ with respect to the new coordinates $\tilde{x}^\mu$?

   (b) A covector field W has components $W_\mu = (t + x, t - x)$ with respect to the old coordinates. What are its components $\widetilde{W}_\mu$ with respect to the new coordinates?

2. In a space-time with metric $ds^2 = dt^2 + 2\,dt\,dx - t^2\,dx^2 - dy^2 - dz^2$, a curve is defined by $t(s) = s$, $x(s) = -s/2$, $y = z = 1$, for $s \in (0, 1)$.

   (a) What is the signature of the metric?

   (b) Is the curve timelike, null or spacelike?

   (c) Compute the proper length L of the curve.

3. A two-dimensional space with coordinates (x^1, x^2) is equipped with a connection, the only nonzero coefficient of which is $\Gamma^2_{11} = x^2$. Find the affinely-parametrized geodesic $x^\mu(s)$ satisfying the initial conditions $x^\mu(0) = (0, 1)$ and $\dot{x}^\mu(0) = (1, 0)$.

4. (a) State (in any of its forms) the condition for a vector field V^μ to be a Killing vector field.

   (b) Show that if V^μ is a Killing vector field and u^μ is the tangent vector to an affinely parametrized geodesic, then the quantity $Q = V^\mu u_\mu$ is constant along the geodesic.

   (c) A *conformal* Killing vector field is a vector field K^μ that satisfies the equation:

   $$\nabla_\mu K_\nu + \nabla_\nu K_\mu = f(x) g_{\mu\nu}$$

   with $f(x)$ some function on spacetime. Show that if K^μ is a conformal Killing vector field and u^μ is the tangent vector to a null geodesic, the quantity $Q' = K^\mu u_\mu$ is constant along the null geodesic.

5. The Friedmann equations describing cosmological dynamics are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) , \qquad \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho .$$

   (a) Determine the time-dependence of the scale factor $a(t)$ for the case of radiation domination with zero spatial curvature.

   (b) Determine the time-dependence of the scale factor $a(t)$ for the case of matter domination with zero spatial curvature.

6. Consider four-dimensional flat space in spherical polar coordinates:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

   (a) How many Killing vector fields does this spacetime have? Write down the Killing vectors associated with conservation of energy and (one component of) angular momentum.

   (b) Use these Killing vectors to write down the massive geodesic equation in the form

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 + V(r) = E,$$

   expressing $V(r)$ and E in terms of constants of the motion, where s is an affine parameter along the geodesic. You may assume the geodesic lies in the equatorial plane with $\theta = \pi/2$. Describe in words the possible solutions to these equations. What form do these trajectories take in Cartesian coordinates (i.e. the coordinates t, x, y, z in which the metric takes the form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$)?

SECTION B

7. Consider the two-dimensional space-time with local coordinates $(x^0, x^1) = (t, x)$ and metric $ds^2 = x\, dt^2 - x^{-1}\, dx^2$ for $x > 0$.

   (a) Find the affinely-parametrized timelike geodesic $x^\mu = x^\mu(s)$ for $|s| < 2$, satisfying the initial conditions $x^\mu = (0, 1)$ and $\dot{x}^\mu = (1, 0)$ at $s = 0$. [You may find the following integral useful: $\int du/(1 - b^2u^2) = b^{-1}\tanh^{-1}(bu)$.]

   (b) Compute the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$.

   (c) Find the vector field V^μ which is parallel-propagated (covariantly-constant) along the curve $x = 1/2$, with $V^\mu = (2, 0)$ at $x^\mu = (0, 1/2)$.

8. (a) Given a metric $g_{\mu\nu}$ and a torsion-free connection such that $\nabla_\alpha g_{\mu\nu} = 0$, derive an expression for the connection coefficients $\Gamma^\alpha_{\mu\nu}$ in terms of $g_{\mu\nu}$.

   (b) A space-time has the metric $ds^2 = dt^2 - e^{2x}(dx^2 + dy^2) - dz^2$. Write out the geodesic equations, and show that the quantity $M = e^x(\dot{x}\cos(y) + p\dot{y}\sin(y))$ is constant along affinely-parametrized geodesics with tangent vector $V^\mu = (\dot{x}, \dot{y})$, for some constant p which you should determine.

   (c) If V^α is a vector field such that $\nabla_\alpha V^\alpha = 0$, derive an expression for $\nabla_\beta \nabla_\alpha V^\beta$ in terms of V^β and the Ricci tensor $R_{\alpha\beta}$.

9. The Friedmann equation for cosmological dynamics with no matter but with a cosmological constant Λ is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{\Lambda}{3}.$$

For this problem some indefinite integrals which may be useful are $\int (1+x^2)^{-1/2}\, dx = \sinh^{-1}(x)$ and $\int (x^2 - 1)^{-1/2}\, dx = \cosh^{-1}(x)$.

   (a) Consider two observers who are at rest with fixed spatial coordinates in an FRW cosmology with zero spatial curvature. One observer emits a photon with frequency ω_1 at $t = t_1$ and it is observed by the other observer at time t_2. Derive an expression for the frequency ω_2 observed at t_2.

   (b) Assume $\Lambda > 0$ and no matter. Assuming an expanding cosmology, determine the time evolution of the scale factor $a(t)$ for positive and negative spatial curvature. Show that for zero spatial curvature we have $a(t) = e^{Ht}$, where H is a constant to be determined.

   (c) For the remainder of this problem, we assume zero spatial curvature and use the solution $a(t) = e^{Ht}$ found in the previous part. In this case, the FRW metric can be written as

$$ds^2 = dt^2 - a(t)^2\left(dx^2 + dy^2 + dz^2\right).$$

   The nonzero Christoffel symbols for this metric are

$$\Gamma^t_{ij} = a\dot{a}\delta_{ij}, \qquad \Gamma^i_{jt} = \frac{\dot{a}}{a}\delta_{ij},$$

   where as usual i, j run over x, y, z.

   Show that a particle trajectory with spatial coordinates fixed (i.e. $x^i(s) = \text{const}$) is a solution to the geodesic equations.

   (d) Consider now a particle with a small initial velocity v in the x direction, i.e. set $y(s) = z(s) = 0$ and $x(s_0) = 0, dx(s_0)/ds = v$, where s is an affine parameter along the geodesic, and s_0 is an initial value of this parameter. Working to first order in $x(s)$, solve the geodesic equations to find $x(s)$ for all $s > s_0$. In particular, show that as $s \to \infty$ the solution $x(s)$ approaches a constant.

10. Consider the metric for an electrically-charged black hole in four dimensions, which takes the form

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right).$$

Here I have set Newton's constant $G \to 1$, and Q is a parameter that is proportional to the electric charge of the black hole.

(a) Consider the case $Q = 0$. Find the curvature and coordinate singularities, if any. Is the singularity at $r = 0$ spacelike or timelike?

(b) Now consider $Q \neq 0$, with $Q^2 < M^2$. Find the curvature and coordinate singularities, if any. Is the singularity at $r = 0$ spacelike or timelike?

(c) Consider $Q \neq 0$, with $Q^2 > M^2$. Find the curvature and coordinate singularities, if any. Is the singularity at $r = 0$ spacelike or timelike?

(d) For this part of the problem, do not assume that we are working with the metric or electric potential above. Consider a particle with mass m and charge q moving on an arbitrary gravitational background with a nonzero electric potential A_μ. If the four-velocity of the particle is u^μ, then the geodesic equation for this particle is modified to become:

$$mu^\mu \nabla_\mu u^\nu = qF^{\mu\nu}u_\mu, \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu.$$

Suppose that ζ is a Killing vector for the metric which also satisfies the further condition

$$\zeta^\sigma \nabla_\sigma A_\nu + (\nabla_\nu \zeta^\sigma)A_\sigma = 0.$$

Show that then the quantity Z defined below is constant on the particle trajectory:

$$Z \equiv \zeta_\mu\left(u^\mu + \alpha A^\mu\right),$$

where α is a constant that should be determined.