

EXAMINATION PAPER

Examination Session: May

2018

Year:

Exam Code:

MATH4051-WE01

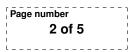
Title:

General Relativity IV

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

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Revision:



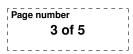
SECTION A

- 1. A two-dimensional space has coordinates $x^{\mu} = (t, x)$. New local coordinates $\tilde{x}^{\mu} = (u, v)$ are defined by $u = t^2 x^2$ and v = 2tx.
 - (a) A vector field V has components $V^{\mu} = (t, x)$ with respect to the old coordinates x^{μ} . What are its components \widetilde{V}^{μ} with respect to the new coordinates \tilde{x}^{μ} ?
 - (b) A covector field W has components $W_{\mu} = (t + x, t x)$ with respect to the old coordinates. What are its components \widetilde{W}_{μ} with respect to the new coordinates?
- 2. In a space-time with metric $ds^2 = dt^2 + 2 dt dx t^2 dx^2 dy^2 dz^2$, a curve is defined by t(s) = s, x(s) = -s/2, y = z = 1, for $s \in (0, 1)$.
 - (a) What is the signature of the metric?
 - (b) Is the curve timelike, null or spacelike?
 - (c) Compute the proper length L of the curve.
- 3. A two-dimensional space with coordinates (x^1, x^2) is equipped with a connection, the only nonzero coefficient of which is $\Gamma_{11}^2 = x^2$. Find the affinely-parametrized geodesic $x^{\mu}(s)$ satisfying the initial conditions $x^{\mu}(0) = (0, 1)$ and $\dot{x}^{\mu}(0) = (1, 0)$.
- 4. (a) State (in any of its forms) the condition for a vector field V^{μ} to be a Killing vector field.
 - (b) Show that if V^{μ} is a Killing vector field and u^{μ} is the tangent vector to an affinely parametrized geodesic, then the quantity $Q = V^{\mu}u_{\mu}$ is constant along the geodesic.
 - (c) A conformal Killing vector field is a vector field K^{μ} that satisfies the equation:

$$\nabla_{\mu}K_{\nu} + \nabla_{\nu}K_{\mu} = f(x)g_{\mu\nu}$$

with f(x) some function on spacetime. Show that if K^{μ} is a conformal Killing vector field and u^{μ} is the tangent vector to a null geodesic, the quantity $Q' = K^{\mu}u_{\mu}$ is constant along the null geodesic.

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5. The Friedmann equations describing cosmological dynamics are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3p\right) \;, \qquad \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho \;.$$

- (a) Determine the time-dependence of the scale factor a(t) for the case of radiation domination with zero spatial curvature.
- (b) Determine the time-dependence of the scale factor a(t) for the case of matter domination with zero spatial curvature.
- 6. Consider four-dimensional flat space in spherical polar coordinates:

$$ds^{2} = dt^{2} - dr^{2} - r^{2}(d\theta^{2} + \sin^{2}\theta \, d\phi^{2}).$$

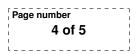
- (a) How many Killing vector fields does this spacetime have? Write down the Killing vectors associated with conservation of energy and (one component of) angular momentum.
- (b) Use these Killing vectors to write down the massive geodesic equation in the form

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 + V(r) = E,$$

expressing V(r) and E in terms of constants of the motion, where s is an affine parameter along the geodesic. You may assume the geodesic lies in the equatorial plane with $\theta = \pi/2$. Describe in words the possible solutions to these equations. What form do these trajectories take in Cartesian coordinates (i.e. the coordinates t, x, y, z in which the metric takes the form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$)?

SECTION B

- 7. Consider the two-dimensional space-time with local coordinates $(x^0, x^1) = (t, x)$ and metric $ds^2 = x dt^2 x^{-1} dx^2$ for x > 0.
 - (a) Find the affinely-parametrized timelike geodesic $x^{\mu} = x^{\mu}(s)$ for |s| < 2, satisfying the initial conditions $x^{\mu} = (0, 1)$ and $\dot{x}^{\mu} = (1, 0)$ at s = 0. [You may find the following integral useful: $\int du/(1 b^2u^2) = b^{-1} \tanh^{-1}(bu)$.]
 - (b) Compute the Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$.
 - (c) Find the vector field V^{μ} which is parallel-propagated (covariantly-constant) along the curve x = 1/2, with $V^{\mu} = (2, 0)$ at $x^{\mu} = (0, 1/2)$.



- 8. (a) Given a metric $g_{\mu\nu}$ and a torsion-free connection such that $\nabla_{\alpha} g_{\mu\nu} = 0$, derive an expression for the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ in terms of $g_{\mu\nu}$.
 - (b) A space-time has the metric $ds^2 = dt^2 e^{2x}(dx^2 + dy^2) dz^2$. Write out the geodesic equations, and show that the quantity $M = e^x(\dot{x}\cos(y) + p\dot{y}\sin(y))$ is constant along affinely-parametrized geodesics with tangent vector $V^{\mu} = (\dot{x}, \dot{y})$, for some constant p which you should determine.
 - (c) If V^{α} is a vector field such that $\nabla_{\alpha}V^{\alpha} = 0$, derive an expression for $\nabla_{\beta}\nabla_{\alpha}V^{\beta}$ in terms of V^{β} and the Ricci tensor $R_{\alpha\beta}$.
- 9. The Friedmann equation for cosmological dynamics with no matter but with a cosmological constant Λ is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{\Lambda}{3}.$$

For this problem some indefinite integrals which may be useful are $\int (1+x^2)^{-1/2} dx = \sinh^{-1}(x)$ and $\int (x^2-1)^{-1/2} dx = \cosh^{-1}(x)$.

- (a) Consider two observers who are at rest with fixed spatial coordinates in an FRW cosmology with zero spatial curvature. One observer emits a photon with frequency ω_1 at $t = t_1$ and it is observed by the other observer at time t_2 . Derive an expression for the frequency ω_2 observed at t_2 .
- (b) Assume $\Lambda > 0$ and no matter. Assuming an expanding cosmology, determine the time evolution of the scale factor a(t) for positive and negative spatial curvature. Show that for zero spatial curvature we have $a(t) = e^{Ht}$, where His a constant to be determined.
- (c) For the remainder of this problem, we assume zero spatial curvature and use the solution $a(t) = e^{Ht}$ found in the previous part. In this case, the FRW metric can be written as

$$ds^{2} = dt^{2} - a(t)^{2} \left(dx^{2} + dy^{2} + dz^{2} \right).$$

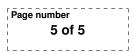
The nonzero Christoffel symbols for this metric are

$$\Gamma^t_{ij} = a\dot{a}\delta_{ij}, \qquad \Gamma^i_{jt} = \frac{\dot{a}}{a}\delta_{ij}$$

where as usual i, j run over x, y, z.

Show that a particle trajectory with spatial coordinates fixed (i.e. $x^i(s) = const$) is a solution to the geodesic equations.

(d) Consider now a particle with a small initial velocity v in the x direction, i.e. set y(s) = z(s) = 0 and $x(s_0) = 0$, $dx(s_0)/ds = v$, where s is an affine parameter along the geodesic, and s_0 is an initial value of this parameter. Working to first order in x(s), solve the geodesic equations to find x(s) for all $s > s_0$. In particular, show that as $s \to \infty$ the solution x(s) approaches a constant.



10. Consider the metric for an electrically-charged black hole in four dimensions, which takes the form

$$ds^{2} = f(r)dt^{2} - \frac{dr^{2}}{f(r)} - r^{2} \left(d\theta^{2} + \sin^{2} \theta \, d\phi^{2} \right),$$

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where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right).$$

Here I have set Newton's constant $G \to 1$, and Q is a parameter that is proportional to the electric charge of the black hole.

- (a) Consider the case Q = 0. Find the curvature and coordinate singularities, if any. Is the singularity at r = 0 spacelike or timelike?
- (b) Now consider $Q \neq 0$, with $Q^2 < M^2$. Find the curvature and coordinate singularities, if any. Is the singularity at r = 0 spacelike or timelike?
- (c) Consider $Q \neq 0$, with $Q^2 > M^2$. Find the curvature and coordinate singularities, if any. Is the singularity at r = 0 spacelike or timelike?
- (d) For this part of the problem, do not assume that we are working with the metric or electric potential above. Consider a particle with mass m and charge q moving on an arbitrary gravitational background with a nonzero electric potential A_{μ} . If the four-velocity of the particle is u^{μ} , then the geodesic equation for this particle is modified to become:

$$mu^{\mu}\nabla_{\mu}u^{\nu} = qF^{\mu\nu}u_{\mu}, \qquad F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}.$$

Suppose that ζ is a Killing vector for the metric which also satisfies the further condition

$$\zeta^{\sigma} \nabla_{\sigma} A_{\nu} + (\nabla_{\nu} \zeta^{\sigma}) A_{\sigma} = 0.$$

Show that then the quantity Z defined below is constant on the particle trajectory:

$$Z \equiv \zeta_{\mu} \left(u^{\mu} + \alpha A^{\mu} \right),$$

where α is a constant that should be determined.