Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4061-WE01

Title: Advanced Quantum Theory IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. The action for two real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{2}m_1^2\varphi_1(x)^2 - \frac{1}{2}m_2^2\varphi_2(x)^2 - \frac{1}{2}\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) - \frac{1}{2}\partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x) + \lambda\Big(\varphi_1(x)^2 + \varphi_2(x)^2\Big)^{3/2} \right\}. \quad (1)$$

(a) Write the equations of motion for the fields φ_1 and φ_2.

(b) Show explicitly that the action S is invariant under the transformation of the fields

$$\begin{aligned} \varphi_1 &\to \cos\alpha\,\varphi_1 + \sin\alpha\,\varphi_2\,, \\ \varphi_2 &\to -\sin\alpha\,\varphi_1 + \cos\alpha\,\varphi_2\,, \end{aligned} \quad (2)$$

where $\alpha = \text{const}$, and derive the Noether current(s) associated with this symmetry.

(c) What is the most general interaction term (which involves only fields and not their derivatives) that one could add to the action S, so that the new action stays invariant under (2)?

(d) Find the new Noether current for the new action which is obtained by adding the general interaction term given in (c) to the original action S.

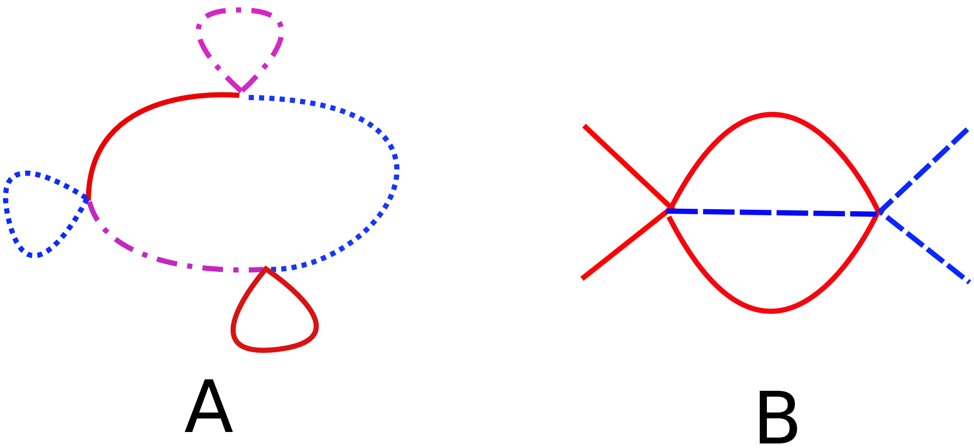
2. (a) For the free complex scalar field, write down the definition of the Feynman propagator which was given in the lectures. Give the physical motivation of this formula. Is this propagator invariant under time inversion $t \to -t$?

(b) Starting with the definition from part (a), show that the Feynman propagator can be written as

$$G_F(x-y) = \theta(y^0 - x^0)[\hat{\phi}_+^\dagger(y), \hat{\phi}_+(x)] + \theta(x^0 - y^0)[\hat{\phi}_-(x), \hat{\phi}_-^\dagger(y)]$$

where $\hat{\phi}_+(x)$ and $\hat{\phi}_-(x)$ are the positive and negative frequency parts of the scalar field $\hat{\phi}(x)$.

3. Figures A and B below show two Feynman graphs which originate from two different theories.



Based on your general knowledge about Feynman graphs answer the following questions:

(a) At which order in perturbation theory do these graphs appear for the first time?

(b) What are the symmetry factors of these graphs?

(c) Write the interaction terms in the Langrangians for the theories from which these graphs originate. Use standard normalisation for the interactions.

(d) Do these graphs contribute to the probability of particles to scatter? Please explain your answer.

4. Consider the generating functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar} S[\phi] + \int d^4x J(x)\phi(x)\right)$$

for a free scalar field theory, with action

$$S[\phi] = -\frac{1}{2}\int d^4x \, \phi\, (\Box + m^2)\phi\, .$$

(a) Rewrite $Z[J]$ in terms of $\tilde{\phi} = \phi + \int d^4y G_F(x-y)J(y)$ in order to complete the square in the exponent. What equation must $G_F(x-y)$ satisfy?

(b) Compute the correlation function

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{Z[0]}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)} Z[J]|_{J=0}$$

from $Z[J]$ and draw the corresponding Feynman diagrams.

5. The action for a massive relativistic point particle in the Polyakov form is

$$S_{\mathrm{P}} = \frac{1}{2}\int d\tau \left[\frac{1}{e(\tau)}\frac{\partial X^\mu(\tau)}{\partial\tau}\frac{\partial X_\mu(\tau)}{\partial\tau} - e(\tau)m^2\right]$$

and in Nambu-Goto form is

$$S_{\mathrm{NG}} = -m\int d\tau\sqrt{-\frac{\partial X^\mu(\tau)}{\partial\tau}\frac{\partial X_\mu(\tau)}{\partial\tau}}\ .$$

[We are using $(-,+,+,\dots)$ signature.]

(a) Compute the equations of motion arising from the Polyakov action.

(b) Compute the equations of motion arising from the Nambu-Goto action and show that they are equivalent to those of the Polyakov action.

(c) A solution of the Polyakov equations has

$$X^0(\tau) = \tau^2/2, \quad X^1(\tau) = a\tau^2/2, \quad X^i(\tau) = 0, \text{ for } i = 2, 3, \dots$$

for some constant a. Use the equations of motion to give $e(\tau)$ in terms of τ, a and m assuming $m > 0$. What does this motion correspond to? What happens when $m \to 0$?

6. ***There is no question 6 on this paper.***

SECTION B

7. A free real scalar field $\phi(x)$ has the action

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}m^2\varphi(x)^2 - \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) \right) \qquad \mu = 0,1,2,3. \tag{3}$$

(a) Write down the equation of motion following from this action, and by explicitly solving this equation, find the most general solution.

(b) Write down the quantum version of the solution found in the previous part and state the commutation relations which the quantum field $\hat{\phi}$ and its conjugate satisfy. Also write down the commutation relations that the operators appearing in the quantum field satisfy.

(c) The action (3) has conserved Noether momenta P_i $(i = 1,2,3)$ given by

$$P_i = \int \mathrm{d}^3x\, \pi(x)\partial_i\varphi(x)\,, \qquad \pi(x) = \partial_0\phi(x)\,.$$

Work out the quantum version of this operator in the normal ordered form and find the normal ordering constant.

(d) Evaluate the following expressions

$$[\hat{P}_i, \hat{\varphi}(x)] \qquad \text{and} \qquad [\hat{P}_i, \hat{\pi}(x)]\,.$$

Express the results using operators $\hat{\pi}$, $\hat{\varphi}$ and their derivatives (not the expansion of these operators).

8. The action for four real scalar fields $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ is given by

$$S = -\int \mathrm{d}^4x \left\{ \frac{1}{2} \left(\sum_{i=1}^{4} \partial_\mu \varphi_i(x) \partial^\mu \varphi_i(x) + \sum_{i=1}^{4} m_i^2 \varphi_i(x)^2 \right) + \lambda \varphi_1(x)\varphi_2(x)\varphi_3(x)\varphi_4(x) \right\}$$

where λ is a real number, a coupling constant.

(a) Write down the Feynman rules for this theory in position and momentum space.

(b) List all the vacuum bubbles which appear in this theory up to and including order λ^4. You should draw all the graphs and write the expressions for these graphs in position space. You do not need to evaluate any of the graphs.

(c) Is there any statement you can make about bubble graphs which is specific to this theory and is not valid in general?

(d) Evaluate the two-point correlator $\langle \Omega | T\{\varphi_1(x)\varphi_2(y)\} | \Omega \rangle$. Start by writing the Dyson formula for two fields. Please explain your answer.

9. ***There is no question 9 on this paper.***

10. The action for a closed string propagating in Minkowski space is

$$S = -\frac{T}{2}\int d\tau \int_{\sigma_L}^{\sigma_R} d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \ .$$

(a) Explain what each symbol in the action means, in words and using a picture.

(b) Derive the equations of motion from this action.

Hint: When deriving the equations of motion for the world sheet metric, you may use without proof that

$$\frac{\partial \sqrt{-h}}{\partial h^{\alpha\beta}} = -\frac{1}{2}\sqrt{-h} h_{\alpha\beta} \, .$$

(c) Show that the following is a solution to the X^μ equations of motion:

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & a^2\sigma^2 \end{pmatrix} \, , \qquad \begin{aligned} X^0 &= \frac{R}{2}\tau \, , \\ X^1 &= R\cos(\sigma^2)\cos\tau \, , \\ X^2 &= R\cos(\sigma^2)\sin\tau \end{aligned}$$

for some constant a you should find.

(d) The solution given above is not in the flat gauge (in which the world-sheet metric equals the Minkowski metric). Perform a world-sheet coordinate transformation on σ to bring the metric to flat form and hence interpret the solution above.