Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4091-WE01

Title:
Stochastic Processes IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Let $(X_n)_{n\geq 0}$, $X_0 = 1$, be a branching process with Binomial $\mathsf{Bin}(m,p)$ offspring distribution, and let $(Y_n)_{n\geq 0}$, $Y_0 = 1$, be a branching process with Poisson $\mathsf{Poi}(\lambda)$ offspring distribution.

   (a) Carefully describe all values of the parameters for which the processes X_n and Y_n are subcritical and for which they are supercritical.

   (b) Let $mp = \lambda$. Which of the two processes is more likely to survive forever? Justify your answer.
   [**Hint:** You may use without proof the following inequality: $1 + x \leq e^x$ for all real x.]

   In your answer you should carefully define branching processes and extinction probabilities, and give a clear statement of any result you use.

2. Let $(X_k)_{k\geq 1}$ be a sequence of independent identically distributed random variables with $\mathsf{P}(X_1 = \pm 1) = 1/2$. Let $(S_n)_{n\geq 0}$ be the simple symmetric random walk generated by $(X_k)_{k\geq 1}$, namely, $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$ for $n > 0$. Define

$$H \stackrel{\mathsf{def}}{=} \inf\{n \geq 0 : S_n = 1\}.$$

   (a) Carefully show that S_n is a martingale and that H is a stopping time for $(S_n)_{n\geq 0}$.

   (b) Show that none of the three versions of the optional sampling theorem can be applied to S_H. In each case carefully show which assumptions of the corresponding theorem are violated.

   In your answer you should carefully define martingales and stopping times, and give a clear statement of any result you use.

   [**Hint:** You can use without proof the following result: $\mathsf{E}s^H = (1 - \sqrt{1 - s^2})/s$.]

3. Let X and Y be discrete random variables with distributions $\mathsf{P}(X = x) = p_x$ and $\mathsf{P}(Y = y) = q_y$.

   (a) Define the maximal coupling of X and Y; check that it has the correct marginal distributions.

   (b) In addition, suppose that X is Bernoulli distributed with parameter r and Y is Poisson distributed with parameter r. Derive the maximal coupling of X and Y; deduce the total variation distance between the distributions of X and Y.

4. (a) Carefully define the Poisson process $\big(N(t)\big)_{t\geq 0}$ with intensity $\lambda > 0$, and state its main properties.

   (b) For fixed $t > 0$, write the formula for the distribution of $N(t)$, ie., for the values of probabilities $\mathsf{P}\big(N(t) = k\big)$ with integer $k \geq 0$. Prove your formula for the case $k = 0$.

   (c) Let fixed t_1, t_2 and t_3 satisfy $0 < t_1 < t_2 < t_3$. Find the conditional probability

$$\mathsf{P}\big(N(t_2) - N(t_1) = 2 \mid N(t_3) = 3\big).$$

5. Let $(X_k)_{k\geq 1}$ be independent random variables, where $X_k \sim \mathsf{Exp}(\lambda_k)$ with $\lambda_k \in (0,\infty)$ for all k. Denote $T = \sum_{k\geq 1} X_k$.

   (a) If $\sum_{k\geq 1} 1/\lambda_k < \infty$, carefully show that $\mathsf{P}(T < \infty) = 1$.

   (b) If $\sum_{k\geq 1} 1/\lambda_k = \infty$, carefully show that $\mathsf{P}(T = \infty) = 1$.

   In your answer you should carefully state any result you use.

6. Let $X_1(t)$ and $X_2(t)$ be continuous-time Markov chains on $\{0,1\}$ with common generator

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}, \qquad \alpha\,,\beta > 0\,.$$

   Assuming that the processes $X_1(t)$ and $X_2(t)$ are independent (ie., $X_1(u)$ and $X_2(v)$ are independent for all non-negative u and v), let $X(t) = X_1(t) + X_2(t)$.

   (a) Show that $X(t)$ is a continuous-time Markov chain and find its generator.

   (b) Write down the forward Kolmogorov equations for $X(t)$.

   (c) Find the stationary distribution of $X(t)$.

SECTION B

7. Let $(X_n)_{n\ge0}$ and $(Y_n)_{n\ge0}$ be two branching processes.

   (a) If $X_0 \le Y_0$ and both processes have the same offspring distribution $(\mathsf{p}_m)_{m\ge0}$, show that X_n is stochastically dominated by Y_n (i.e., $X_n \preccurlyeq Y_n$) for all $n \ge 0$. Deduce that the Y-process is more likely to survive than the X-process.

   (b) Suppose now $X_0 = Y_0 = 1$, but the offspring distribution of the X-process is stochastically smaller than the offspring distribution of the Y-process, equivalently, $X_1 \preccurlyeq Y_1$. Show that X_n is stochastically dominated by Y_n for all $n \ge 0$. Deduce that the Y-process is more likely to survive than the X-process.

8. A standard fair die is tossed repeatedly. Let T be the number of tosses until the sequence 1-2-3-3-2-1 is observed for the first time. Use the appropriate optional stopping theorem to find the expectation $\mathsf{E}T$.
   In your answer you should clearly state and carefully apply any result you use.

9. Let $X(t)$, $t \ge 0$, be a continuous-time Markov chain on the state space $\{0,1,2\}$ whose generator (Q-matrix) is

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

   (a) Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t)$, i, $j \in \{0,1,2\}$.

   (b) Show that

$$P(t) = \exp\{tQ\} = \sum_{k\ge0} \frac{t^k}{k!}\, Q^k$$

   is a unique solution to the equations you obtained in part (a).

   (c) Define the resolvent $R(\lambda)$ for $X(t)$ and find

$$p_{12}(t) = \mathsf{P}\big(X(t) = 2 \mid X(0) = 1\big)\,.$$

10. (a) Carefully define a Brownian motion, state its Markov and strong Markov properties.

   (b) Let $(B_t)_{t\ge0}$ be a Brownian motion starting at the origin, $B_0 = 0$, and let T_a be the hitting time of $a > 0$, ie., $T_a = \inf\{t \ge 0 : B_t = a\}$. Use the reflection principle to show that

$$\mathsf{P}(T_a \le t) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t}\, dx\,,$$

   and find the probability density function of T_a.
   [**Hint:** You might wish to change the variables $x \mapsto s$ with $x = t^{1/2}a/s^{1/2}$.]

   (c) Carefully show that for all $0 \le s < t$ we have

$$\mathsf{E}\big(B_t | B_r, 0 \le r \le s\big) = B_s\,, \qquad \mathsf{E}\big(B_t^2 - t | B_r, 0 \le r \le s\big) = B_s^2 - s\,,$$

   ie., that B_t and $B_t^2 - t$ are martingales with respect to the natural filtration.

SECTION C

11. (i) Carefully define a renewal process.

(a) Let $N(t)$ be a renewal process generated by interarrival times $(\tau_k)_{k\geq1}$. If $0 < \mathsf{E}\tau < \infty$, show that, with probability one,

$$\frac{N(t)}{t} \to \frac{1}{\mathsf{E}\tau} \quad \text{as } t \to \infty\,.$$

(b) For the renewal process $N(t)$ as above let a reward ρ_k be earned at the time $T_k = \tau_1 + \cdots + \tau_k$ of the kth renewal. Assuming that $\mathsf{E}\rho < \infty$, and that the pairs (τ_k, ρ_k) are independent and have the same distribution, let $\mathcal{R}(t) = \sum_{k=1}^{N(t)} \rho_k$ be the total reward earned by time t. Show that, with probability one,

$$\frac{\mathcal{R}(t)}{t} \to \frac{\mathsf{E}\rho}{\mathsf{E}\tau} \quad \text{as } t \to \infty\,.$$

(ii) Carefully define an alternating renewal process.

Suppose that a machine works for time w_1 and then fails; the subsequent repair lasts time r_1 and brings the machine back to a "like new" condition. The process repeats with times w_2, r_2, w_3, r_3 and so on. Assuming that the pairs (w_k, r_k) are independent and that $\mathsf{E}w$ and $\mathsf{E}r$ are both finite, describe the state of the machine as an alternating renewal process. Show that, with probability one, the fraction of time the machine spends working converges to $\mathsf{E}w/(\mathsf{E}w + \mathsf{E}r)$.