Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4121-WE01

Title: Solitons IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Compute the dispersion relation for the equation

$$u_t + u_x - a u_{xx} - b u_{xxx} = 0$$

where a and b are real constants.

(a) For which values of a and b is there dissipation? And for which of these values is the dissipation physical? (Recall that a wave has physical dissipation if the amplitude of the wave decreases with time.)

(b) For $a = 0$, compute the phase and group velocities $c(k)$ and $c_g(k)$, and discuss for which values of b there is dispersion.

2. Construct a travelling wave solution with velocity $v > 0$ of the equation

$$u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0 \ ,$$

subject to the boundary conditions $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$, where n is a positive integer. You can assume that $0 < u \leq (v/2)^{1/n}$ and use the indefinite integral

$$\int \frac{dx}{x\sqrt{1-x}} = -2\mathrm{arctanh}(\sqrt{1-x}) + \text{constant}$$

without proof.

3. (a) Given an equation of motion and boundary conditions for the field $u(x,t)$, state conditions under which a charge

$$Q = \int_{-\infty}^{+\infty} dx \ \rho(u, u_t, u_x, \ldots)$$

is conserved.

(b) Show that for the equation

$$u_t + 12u^2 u_x + u_{xxx} = 0$$

with boundary conditions $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$, the charge densities $\rho_1 = u$ and $\rho_2 = u^2$ satisfy the conditions stated in part (a) and therefore lead to two conserved charges $Q_i = \int_{-\infty}^{+\infty} dx \ \rho_i$, with $i = 1, 2$.

4. The equation $-d^2\psi/dx^2 + V(x)\psi = -\lambda\psi$ has a solution of the form

$$\psi(x) = e^{ikx}(ik - \tanh(x)) \ ,$$

where $\lambda = -k^2$.

(a) Find $V(x)$. [Hint: substitute the solution $\psi(x)$ into the equation.]

(b) By considering the asymptotic behaviour of $\tanh(x)$ and by normalising ψ appropriately, find the reflection and transmission coefficients for this solution.

(c) For what value(s) of λ is there a bound state solution?

5. (a) Define the Hirota differential operator $D_t^m D_x^n(f, g)$ acting on a pair of functions f and g. Compute $D_x^4(f, g)$, and show that

$$D_x^4(f, f) = 2(f f_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) \, .$$

(b) In the ball and box model, space is replaced by an infinite line of boxes, indexed by an integer $k \in \mathbb{Z}$. At time $t = 0$ there are balls in boxes 1, 2, 3, 7 and 8, and all the other boxes are empty. Evolve this configuration to $t = 1$ and 2 using the ball and box rule, and determine the phase shift of the length-3 soliton.

6. ***There is no question 6 on this paper.***

SECTION B

7. A field $u(x,t)$ has energy given by

$$E[u] = \frac{1}{2}\int_{-\infty}^{+\infty} dx \ \left[u_t^2 + u_x^2 + W(u)^2\right] \ ,$$

where $W(u)$ is a real function of u.

(a) Which boundary conditions on u, u_x and u_t should be satisfied as $x \to \pm\infty$ in order for the field u to have finite energy?

(b) The function $W(u)$ is such that $W(u_-) = W(u_+) = 0$ and $W(u) > 0$ for $u_- < u < u_+$. If the field $u(x,t)$ is smooth and satisfies finite energy boundary conditions with $u \to u_\pm$ as $x \to \pm\infty$, prove the Bogomol'nyi bound $E[u] \geq K$, where K is a positive constant which you should relate to $W(u)$. Which equations should the field u satisfy if it saturates the bound?

(c) Compute K and find the most general solution that saturates the Bogomol'nyi bound if $W(u) = \cos^2(u)$ with $u_\pm = \pm\frac{\pi}{2}$. Check explicitly that the boundary conditions are satisfied.

8. Consider the pair of equations

$$v_x = -\frac{1}{2}uv \ , \qquad v_t = \frac{1}{4}(u^2 - 2u_x)v \ .$$

(a) Show that these relations give a Bäcklund transform between Burgers' equation $u_t + uu_x - u_{xx} = 0$ for u and the heat equation $v_t = v_{xx}$ for v.

(b) $u(x,t) = 2c$ is a solution of Burgers' equation, where c is a constant. Apply the Bäcklund transform to find the corresponding solution of the heat equation.

(c) A special solution of the heat equation for $t > 0$, which can be obtained by time evolution of a Dirac delta function at $t = 0$, is

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-x^2/(4t)) \ .$$

[You do not need to check these statements.] Apply the Bäcklund transform to find the corresponding solution of Burgers' equation. Is this solution of Burgers' equation regular or singular?

9. (a) If $f = f(x,t)$, $D := \partial/\partial x$, and P, Q, R are any (differential) operators, show that:

   (i) $[D, f] = f_x$,

   (ii) $[D^2, f] = f_{xx} + 2f_x D$,

   (iii) $[D^3, f] = f_{xxx} + 3f_{xx}D + 3f_x D^2$,

   (iv) $[P, QR] = [P, Q]R + Q[P, R]$.

   (b) Let L, M be the differential operators $L = D^2 + u(x,t)$, and $M = -4D^3 + \beta(x,t)D + \gamma(x,t)$. Compute the commutator $[L, M]$, writing it in a form where the differential operators D^n are all on the right.

   (c) What property must the commutator $[L, M]$ possess in order for L, M to form a Lax pair? Use this property to find $\beta(x,t)$ and $\gamma(x,t)$ in terms of two unknown functions of t only.

   (d) Solve these equations and show that the Lax equation, $L_t + [L, M] = 0$, implies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for a particular choice of one of the unknown functions of t.

   (e) Indicate how one can generalise this method to obtain higher order non-linear partial differential equations for u which are solvable via the inverse scattering method.

10. (a) If $F[u]$ is the functional $F[u] = \int_{-\infty}^{+\infty} dx\; f(u, u_x, u_{xx}, \dots)$, define the functional derivative $\delta F[u]/\delta u$, and derive an expression for this derivative in terms of $\partial f/\partial u$, $\partial f/\partial u_x$, $\partial f/\partial u_{xx}$ etc. (The function u satisfies the boundary conditions $u \to 0$, $u_x \to 0$, $u_{xx} \to 0$, etc as $x \to \pm\infty$.)

   (b) Take $f(u, u_x, \dots) = au_x^2 + bu^3$, for a, b constants, and write out the differential equation
   $$u_t = \frac{\partial}{\partial x}\left(\frac{\delta F[u]}{\delta u}\right).$$
   Find the values of the constants a, b for which this becomes the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

   (c) Now take $f(u, u_x, \dots) = \alpha u^4 + \beta u u_x^2 + \gamma u_{xx}^2$, for α, β, γ constants, and write out the differential equation
   $$u_t = \frac{\partial}{\partial x}\left(\frac{\delta F[u]}{\delta u}\right).$$
   Find the values of the constants α, β, γ for which this becomes the KdV_5 equation $u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0$.

   (d) Explain the significance of the results in parts (b) and (c).

SECTION C

11. A field $u(x,t)$ is defined on the half-line $x \in (-\infty, 0]$, and has kinetic and potential energy

$$T = \int_{-\infty}^{0} dx \, \frac{1}{2} u_t^2 \ , \qquad V = \int_{-\infty}^{0} dx \left[\frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] + B(u(0,t)) \ ,$$

where B is a function of the value of the field at the $x = 0$ boundary.

(a) Which boundary conditions should be imposed on u, u_x and u_t as $x \to -\infty$ to ensure that the total energy $E = T + V$ is finite?

(b) Write down the action S. Use the principle of least (or stationary) action to derive the bulk equation of motion as well as the boundary condition

$$u_x(0,t) = -B'(u(0,t))$$

for the field at $x = 0$.

(c) Now assume that the boundary contribution to the potential energy is

$$B(u(0,t)) = \mu (u(0,t) - u_0)^2 \ ,$$

where $\mu > 0$ and u_0 are constants. Write down the boundary condition for the field at $x = 0$. What is this boundary condition called? For which values of the constants μ and u_0 does it reduce to the 'free' (or Neumann) boundary condition? And to the 'fixed' (or Dirichlet) boundary condition?

(d) Assuming that $B(u(0,t))$ takes the form introduced in part (c), show that the total energy is conserved if the bulk equations of motion and the boundary conditions are satisfied.