EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4161-WE01

Title:
Algebraic Topology IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

University of Durham Copyright

The following notations hold in this paper

- $\mathbb{R}^n$ denotes real n-space with the usual topology.
- D^{n+1} and S^n denote the closed unit ball and unit sphere in $\mathbb{R}^{n+1}$ with the subspace topology.
- For n a positive integer, $\mathbb{Z}/n$ denotes the quotient group $\mathbb{Z}/n\mathbb{Z}$.
  Elements of $\mathbb{Z}/n$ are denoted as $\bar{k}$ for $k \in \mathbb{Z}$.

SECTION A

1. (a) What does it mean for two continuous maps to be *homotopic*? What does it mean for two spaces to be *homotopy equivalent*?

   (b) Suppose that F is a functor from the category of topological spaces and continuous maps to some category $\mathcal{C}$. Moreover, assume that F is a homotopy invariant functor, in the sense that if two maps f and g are homotopic then $F(f) = F(g)$. Prove that F maps homotopy equivalent spaces to isomorphic objects in $\mathcal{C}$.

   (c) We say that a space X is *contractible* if the identity map id_X on X is homotopic to the constant map c_{x_0} for some $x_0 \in X$, where $c_{x_0} \colon X \to X$ is defined by $c_{x_0}(x) = x_0$ for all $x \in X$. Show that X is contractible if and only if X is homotopy equivalent to the one point space.

2. (a) Define the relative singular homology of a topological pair (X, A). You need not define the non-relative singular chain complex of a space.

   (b) Write down the long exact sequence of the pair. Given $x \in H_n(X, A)$ represented by a cycle σ, write down an expression for the image of x under the connecting map of this long exact sequence, briefly explaining any notation you use.

   (c) Suppose that X is a CW complex with only even dimensional cells, and that A is a subcomplex of X. Prove that the relative homology groups $H_n(X, A)$ are free abelian. What can we say about the ranks of the relative homology groups $H_n(X, A)$?

3. (a) What is the *suspension* ΣX of a space X?

   (b) Use the Mayer–Vietoris sequence to prove that $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$ for all $n \in \mathbb{Z}$.

   (c) Prove that $\mathbb{R}\mathbf{P}^2$ is not homeomorphic to ΣX for any space X.

4. Let X be a finite CW-complex consisting of one 0-cell, one 5-cell, and one 6-cell attached via a map $\chi \colon S^5 \to X^5 = S^5$ of degree 25.

   (a) Calculate $H^*(X; \mathbb{Z}/n)$ for $n \geq 2$ satisfying $\gcd(n, 5) = 1$.

   (b) Calculate $H^*(X; \mathbb{Z}/n)$ for $n \geq 2$ satisfying $\gcd(n, 5) = 5$.

University of Durham Copyright

5. Let A, B be abelian groups.

   (a) State the definition of $\mathrm{Ext}(A, B)$.

   (b) Using this definition, calculate $\mathrm{Ext}(\mathbb{Z}/245, \mathbb{Z}/49)$.

6. ***There is no question 6 on this paper.***

University of Durham Copyright

SECTION B

7. (a) State Hurewicz's Theorem relating the homology and fundamental group of a path-connected space. Describe how the map in the statement of the theorem is constructed.

   (b) Let (X, A) be a good pair, where $A \cong S^1$ and X is simply connected (recall that this means that X is path-connected and has trivial fundamental group). What can we say about $H_n(X/A)$ in terms of $H_n(X)$?

   (c) Dropping the condition that X is simply connected in the above, prove by examples that $H_n(X/A)$ may depend on how $A \cong S^1$ is embedded in X.

8. (a) What is meant by a *chain map* and a *chain homotopy* between two chain maps?

   (b) Show that if the identity chain map on a chain complex C_* is chain homotopic to the zero chain map (the chain map which sends each chain to the zero element), then $H_n(C) \cong 0$ for all $n \in \mathbb{Z}$.

   (c) Suppose that each chain group C_n of C_* is a free abelian group, and that $C_n \cong 0$ for $n < 0$. Prove that if $H_n(C) \cong 0$ for all $n \in \mathbb{Z}$ then the identity chain map is chain homotopic to the zero chain map.

   (d) Find a chain complex C_* with C_1 non-trivial, $C_3 \cong \mathbb{Z}^2$ and with the identity chain map chain homotopic to the zero chain map on C_*.

University of Durham Copyright

9. Let X be a topological space, $A \subset X$ and $C_*(X, A)$ the singular chain complex of the pair (X, A). For $\sigma\colon \Delta^n \to X$ and $i \in \{0, \dots, n\}$ let $F_i\sigma\colon \Delta^{n-1} \to X$ be the i-th face of σ. Then define $\bar{\partial}\colon C_n(X, A) \to C_{n-1}(X, A)$ by

$$\bar{\partial}\sigma = \sum_{i=0}^{n} F_i\sigma$$

   (a) Calculate $\bar{\partial}^4(\sigma) = \bar{\partial} \circ \bar{\partial} \circ \bar{\partial} \circ \bar{\partial}(\sigma)$.

   (b) Let $C^n(X, A; \mathbb{Z}/4) = \mathrm{Hom}(C_n(X, A), \mathbb{Z}/4)$ and

$$\bar{\delta}\colon C^n(X, A; \mathbb{Z}/4) \to C^{n+1}(X, A; \mathbb{Z}/4)$$

   be given by

$$\bar{\delta}\varphi = \varphi \circ \bar{\partial}.$$

   Show that $\bar{\delta}^4 = 0$.

   (c) Define

$$\begin{aligned}\bar{H}^n(X, A; \mathbb{Z}/4) = \ker(\bar{\delta}^2\colon C^n(X, A; \mathbb{Z}/4) \to C^{n+2}(X, A; \mathbb{Z}/4))/\\ \mathrm{im}(\bar{\delta}^2\colon C^{n-2}(X, A; \mathbb{Z}/4) \to C^n(X, A; \mathbb{Z}/4)).\end{aligned}$$

   Show that there is a long exact sequence

$$\begin{aligned}\cdots \longrightarrow \bar{H}^n(X; \mathbb{Z}/4) \longrightarrow \bar{H}^n(A; \mathbb{Z}/4) \longrightarrow \bar{H}^{n+2}(X, A; \mathbb{Z}/4) \longrightarrow\\ \longrightarrow \bar{H}^{n+2}(X; \mathbb{Z}/4) \longrightarrow \cdots\end{aligned}$$

   State any results from the lectures that you use.

   (d) Calculate $\bar{H}^n(P; \mathbb{Z}/4)$ for all $n \geq 0$, where P consists of exactly one point.

10. ***There is no question 10 on this paper.***

University of Durham Copyright