EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4171-WE01

Title: Riemannian Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Let X, Y be two vector fields on $\mathbb{R}^3$ defined by

$$\begin{aligned} X(x,y,z) &= 2y\frac{\partial}{\partial x} + (z-2x)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ Y(x,y,z) &= -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}. \end{aligned}$$

   (a) Calculate the Lie bracket $[X, Y]$.

   (b) Let $r > 0$ and $S^2(r) = \{(x,y,z) \mid x^2+y^2+z^2 = r^2\}$ be the sphere of radius r. Show that the restriction of $[X, Y]$ to $S^2(r)$ is a vector field on $S^2(r)$.

2. Let $M := \{(x,y,z) \mid x^2+y^2 = z\}$ be the paraboloid. Choose a suitable coordinate chart $\varphi = (r, \alpha) : M\backslash\{(0,0,0)\} \to (0,\infty) \times (0, 2\pi)$ and calculate

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \alpha}, \quad \nabla_{\frac{\partial}{\partial \alpha}}\frac{\partial}{\partial r}, \quad \nabla_{\frac{\partial}{\partial \alpha}}\frac{\partial}{\partial \alpha}.$$

3. Let (M, g) be a Riemannian manifold.

   (i) Give the definition of the exponential map at a point $p \in M$.

   (ii) Give the statement of the Gauss Lemma at a point $p \in M$.

   (iii) Let $J : [a,b] \to TM$ be a Jacobi field along a geodesic $c : [a,b] \to M$ and $f(t) = \|J(t)\|^2$. Show that if M has non-positive sectional curvature then $f''(t) \geq 0$.

4. (a) Define the Levi-Civita connection on a Riemannian manifold.

   (b) Derive *Bianchi's first identity*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

   from *Jacobi's identity* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$. **You don't need to prove Jacobi's identity.**

5. The Heisenberg group is given by

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

H has a global coordinate chart $\varphi : H \to \mathbb{R}^3$ via

$$\varphi(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) = (x, y, z).$$

Let $e = \varphi^{-1}(0,0,0)$, $g = \varphi^{-1}(x,y,z)$, and $L_g, R_g : H \to H$ be the left- and right-multiplication maps, defined by $L_g(h) = gh$ and $R_g(h) = hg$. Calculate the tangent vectors

$$DL_g(e)\left(\frac{\partial}{\partial y}|_e\right) \in T_gH \quad \text{and } DR_g(e)\left(\frac{\partial}{\partial z}|_e\right) \in T_gH$$

in terms of $\frac{\partial}{\partial x}|_g, \frac{\partial}{\partial y}|_g, \frac{\partial}{\partial z}|_g$.

6. (i) Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map, $n > k$, $y \in f(\mathbb{R}^n)$ be a regular value, $M := f^{-1}(y)$ and $p \in M$. Show that

$$T_pM = \ker Df(p).$$

(ii) Let $Z = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z \in (0,1)\}$ be a cylinder and $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, z \in (0,1)\}$ be a cone. Let $f : Z \to C$ be defined by $f(x,y,z) = (zx, zy, z)$. Check that

$$p = \left(1, 0, \frac{1}{2}\right) \in Z \quad \text{and } v = \left(0, -1, \frac{1}{3}\right) \in T_pZ$$

and calculate $Df(p)(v) \in T_{f(p)}C$.

SECTION B

7. Let $M(n,\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries, $\mathrm{Sym}(n) = \{A \in M(n,\mathbb{R}) \mid A^\top = A\}$ and

$$J = \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \in M(n,\mathbb{R}),$$

where Id_k denotes the identity matrix on $\mathbb{R}^k$. Let

$$O(n-1,1) = \{A \in M(n,\mathbb{R}) \mid AJA^\top = J\}.$$

(a) Show that all matrices in $O(n-1,1)$ are invertible and that $O(n-1,1)$ forms a group under matrix multiplication.

(b) Let $f : M(n,\mathbb{R}) \to \mathrm{Sym}(n)$ be defined as $f(A) = AJA^\top$. Show for $A, B \in M(n,\mathbb{R})$ that

$$Df(A)(B) = AJB^\top + BJA^\top,$$

and conclude that for $A \in O(n-1,1)$ and $C \in \mathrm{Sym}(n)$,

$$Df(A)\left(\frac{1}{2}CJA\right) = C.$$

(c) Show that $J \in \mathrm{Sym}(n)$ is a regular value of f.

(d) Show that $O(n-1,1)$ is differentiable manifold and determine its dimension.

8. (a) Let R be the curvature tensor of a Riemannian manifold (M,g). State the symmetry properties of R. Show that

$$R(X,Y)(fZ) = fR(X,Y)Z$$

for $X,Y,Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

(b) Assume there exists a constant $C \in \mathbb{R}$ such that the curvature tensor R of (M,g) satisfies, for all $X,Y,W,Z \in \mathfrak{X}(M)$,

$$\langle R(X,Y)W,Z\rangle = C(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle). \qquad (*)$$

Show that (M,g) is then an Einstein manifold, that is, $\mathrm{Ric} = \lambda g$ for some function $\lambda \in C^\infty(M)$.

(c) Let (M,g) be a Riemannian manifold and $c : [a,b] \to M$ a geodesic. Let $X \in \mathfrak{X}_c(M)$ be a parallel vector. Show that $t \mapsto \langle X(t), c'(t)\rangle$ is constant.

(d) Let (M,g) be a Riemannian manifold satisfying $(*)$ with a constant $C \leq 0$. Let $c : [a,b] \to M$ be a geodesic parametrised by arc-length and $X \in \mathfrak{X}_c(M)$ be a parallel vector field along c with $X(a) \perp c'(a)$. Show that then the vector field

$$J(t) = \cosh(\sqrt{-C}t)X(t)$$

is a Jacobi field along c.

9. Let $\varphi^{-1}(x,y) = (f(x)\cos(y), f(x)\sin(y), h(x))$ with $f : [a,b] \to (0,\infty)$ and let $h : [a,b] \to \mathbb{R}$ be a coordinate chart of a surface of revolution, denoted by S. Assume that S carries the induced metric of the Euclidean $\mathbb{R}^3$. Assume, furthermore, that

$$(f'(x))^2 + (h'(x))^2 = 1 \quad \text{for all } x \in [a,b].$$

In this question, you may use without proof the following equalities:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= 0, \\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} &= \frac{f'(x)}{f(x)} \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= -f(x)f'(x) \frac{\partial}{\partial x}. \end{aligned}$$

(a) Calculate the first fundamental form $(g_{ij}(x,y))$ of this coordinate chart and show that, for any fixed y_0, the curves $c : [a,b] \to S$, defined by $c(t) = \varphi^{-1}(t, y_0)$, are geodesics and parametrised by arc-length.

(b) Let $R_p : T_pM \times T_pM \times T_pM \to T_pM$ be the curvature tensor of a general Riemannian manifold (M,g) and $\Pi \subset T_pM$ a 2-plane spanned by $v, w \in T_pM$. Give the formula for the sectional curvature $K_p(\Pi)$ in terms of v, w.

(c) Consider the geodesic variation $F : [y_1, y_2] \times [a,b] \to S$, $F(s,t) = \varphi^{-1}(t,s)$ of $c(t) = \varphi^{-1}(t, y_0)$ with $y_0 \in [y_1, y_2]$ and $X \in \mathfrak{X}_c(S)$ its variational vector field. Calculate $X'' = \frac{D^2}{dt}X$ explicitly. Give an explanation why X is a Jacobi field.

(d) Let $p = \varphi^{-1}(x,y)$ with $(x,y) \in [a,b] \times [y_1, y_2]$. Use the explicit formula for X'' and the Jacobi equation $X'' + R(X, c')c' = 0$ to calculate the sectional curvature $K_p(T_pS)$.

10. Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the hyperbolic plane with the identity as global coordinate chart and first fundamental form given by

$$(g_{ij}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

We associate to $A \in \mathrm{SL}(2, \mathbb{R})$ the map $f_A : \mathbb{H}^2 \to \mathbb{H}^2$ via

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow f_A(z) = \frac{az + b}{cz + d},$$

by identifying $\mathbb{H}^2$ with $\mathbb{C} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$.

(a) Show that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

(b) Use the formula $Df_A(z)(v) = \frac{1}{(cz+d)^2} v$ **(without proof)** to show that, for all $A \in \mathrm{SL}(2, \mathbb{R})$, $z \in \mathbb{H}^2$, $v \in T_z\mathbb{H}^2$:

$$\langle Df_A(z)v, Df_A(z)v \rangle_{f_A(z)} = \langle v, v \rangle_z,$$

that is, f_A is an isometry.

(c) Show for $A, B \in \mathrm{SL}(2, \mathbb{R})$ that

$$f_{AB} = f_A \circ f_B.$$

(d) Assume **without proof** that the non-zero Christoffel symbols of the above global coordinate chart at $z = x + iy$ are

$$\Gamma^2_{22}(z) = \Gamma^1_{12}(z) = \Gamma^1_{21}(z) = \frac{-1}{y}, \quad \Gamma^2_{11}(z) = \frac{1}{y}.$$

Show that $\mathbb{H}^2$ has constant sectional curvature -1.