Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4181-WE01

Title:
Mathematical Finance IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Consider a market that has an interest rate of 3%, continuously compounded, and a stock with a current share price of 70. Suppose that the market prices of three European call options on the stock, with expiry time 18 months, are as follows:

Strike price	Current price of option
40	32
60	16
80	6

Suppose also that a strangle option on the stock with the same expiry time is offered on the market, with contract function $\Phi(x) = \max\{|x-70|-10, 0\}$.

(a) Draw the graph of the payoff of the strangle option against the share price of the underlying stock at expiry.

(b) Calculate the current price of the strangle option that avoids arbitrage on the market.

(c) Under what market conditions would you consider purchasing this strangle option?

2. Consider the one-period financial market $\mathcal{M} = (B_t, S_t)$ on two assets, where the risk-free asset price satisfies $B_0 = 1, B_1 = 1 + r$ and the risky asset price satisfies $S_0 = s$,

$$S_1 = \begin{cases} su & \text{with probability } p_u \\ sd & \text{with probability } p_d \end{cases}$$

where $r > 0, s > 0, u > d > 0$ are constant parameters and p_u and p_d are positive probabilities that sum to 1.

(a) Prove that the market is arbitrage free if and only if $d < 1 + r < u$.

(b) Assuming the market is arbitrage free, calculate the no-arbitrage price at time 0 of a contingent claim that returns $+1$ if $S_1 = su$ and -1 if $S_1 = sd$.

3. (a) State the Cox–Ross–Rubinstein formula for the price at time $T - t$ of a European call option with strike price K and expiry time T. What assumptions about the underlying stock behaviour are necessary for the formula to be valid?

(b) Use the formula to find the price at time 0 of a European call option with strike price 70 and expiry date 5 on a stock whose initial share price $S_0 = 100$ and which evolves according to a binomial model with parameters $u = 1.1, d = 0.9, p_u = 2/3, p_d = 1/3$ and interest rate $r = 0.05$.

4. (a) State the Black–Scholes formula for the price C of a European call option on an underlying stock whose price follows a geometric Brownian motion. Define clearly any notation you use.

   (b) Calculate $\Delta = \dfrac{\partial C}{\partial S}$, the derivative with respect to the current share price, and show that $0 \leq \Delta \leq 1$.

5. (a) State the definition of a Brownian motion.

   (b) Suppose $(W_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are independent Brownian motions, and define $X_t = \alpha W_t + \beta V_t$. For which values of α and β is $(X_t)_{t\geq 0}$ a Brownian motion?

   (c) Suppose that $(W_t)_{t\in[0,T]}$ is a Brownian motion on the time interval $[0,T]$. Prove that $(W_T - W_{T-t})_{t\in[0,T]}$ is also a Brownian motion on the time interval $[0,T]$.

6. Let $(W_t)_{t\geq 0}$ be a Brownian motion.

   (a) Give the distribution of the Itô integral $I = \int_0^T f(t)\mathrm{d}W_t$ for a deterministic function $f(t)$.

   (b) State the Itô isometry for a general Itô integral.

   (c) Calculate $\mathbb{E}[\int_0^1 \mathrm{e}^{W_t}\,\mathrm{d}W_t]$ and $\mathbb{V}\mathrm{ar}[\int_0^1 \mathrm{e}^{W_t}\,\mathrm{d}W_t]$.

SECTION B

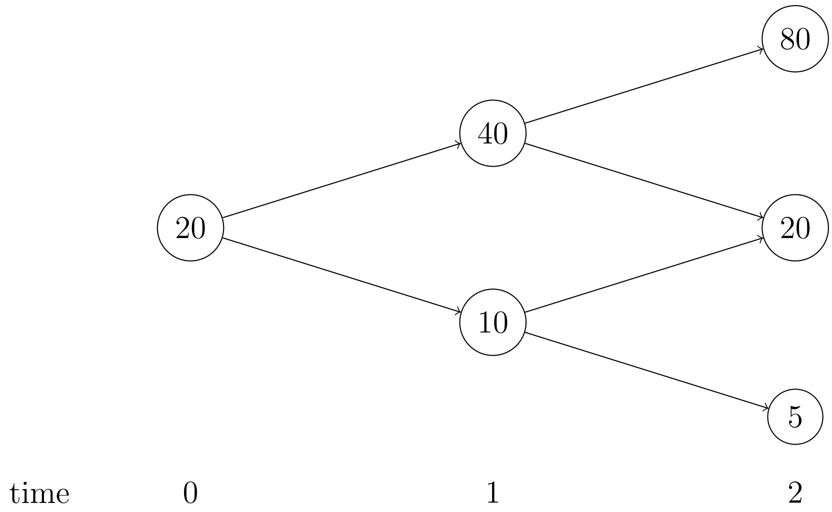
7. (a) State the definitions of a standard lookback call option and standard lookback put option on an underlying asset.

   (b) Suppose the current share price of the asset is $S_0 = 40$ and evolves according to the binomial model with $u = 5/4$, $d = 1/2$, $p_u = 3/4$, $p_d = 1/4$ and that the interest rate is $r = 1/8$. Find the no-arbitrage prices at all times $t = 0, 1, 2, 3$ of both a standard lookback call option and a standard lookback put option on this asset with expiry time $T = 3$.

   (c) A new "chooser option" is offered on the market based on the above options. The chooser option is sold at time 0 and at time 1 the holder must decide whether the option will be a lookback call option, or a lookback put option. What is the price at time 0 of this chooser option?

   (d) Calculate the self-financing portfolio that replicates the option in part (c), in the case where the asset price increases at every time step.

8. Consider a 2-period financial market with possible share prices $S_t, t = 0, 1, 2$, of a risky asset given by the recombining tree:



Suppose the interest rate per time step is $r = \frac{1}{4}$.

(a) A European put option has a value of 2 at time 0. What is its strike price K_E?

(b) An American put option has a value of 2 at time 0. What is its strike price K_A? (*Hint*: You might find it helpful to first explain which of $K_A \leq K_E$ or $K_A \geq K_E$ holds.)

(c) What is the largest strike price K for which the values at time 0 of the European and American put options are the same?

9. (a) State Itô's Lemma for a smooth function $f(t, R_t)$ of time t and an Itô process $(R_t)_{t\geq 0}$.

The Cox–Ingersoll–Ross model for the interest rate process $(R_t)_{t\geq 0}$ is given by

$$\mathrm{d}R_t = (\alpha - \beta R_t)\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}W_t$$

where α, β and σ and R_0 are positive constants, and $(W_t)_{t\geq 0}$ is a Brownian motion.

(b) Calculate $\mathbb{E}[R_t]$ and $\mathbb{V}\mathrm{ar}[R_t]$.
Hint: Applying Itô's Lemma to an appropriate function $f(t, R_t)$ may simplify the calculation.

10. Suppose that a stock has a current price of £15, and moves as a geometric Brownian motion with drift $\mu = 0.1$ and volatility $\sigma = 0.2$, and suppose that the risk-free interest rate is 5% compounded continuously.

(a) Write down the continuous-time Black–Scholes differential equations that model the price dynamics of this market.

(b) Define clearly the risk-neutral measure, and state the risk-neutral valuation formula for a contingent claim X on the underlying stock.

(c) What is the value at time t of a cash-or-nothing binary option which pays off £1 if the share price is more than £15 in one year's time?

(d) Calculate the self-financing portfolio that replicates the option in part (c).

SECTION C

11. (a) Write down an approximate 95% confidence interval for $\mathbb{E}X$ in terms of the usual unbiased estimators for $\mathbb{E}X$ and $\mathbb{V}\mathrm{ar}X$, based on a sample of size M. Justify your expression by appealing to the Central Limit Theorem.

    [You may find some of the following values of the standard normal cdf useful.]

z	1.28	1.645	1.96
$N(z)$	0.9	0.95	0.975

(b) Describe carefully a Monte Carlo algorithm for producing an approximate 95% confidence interval for the fair price at time 0 of a European option on an underlying risky asset with contract function $\Phi(S_T)$. State clearly any assumptions you make about the price of the underlying asset.

(c) Let $\widehat{C}$ and $\widehat{P}$ be the unbiased estimators for the prices of a call option and put option with the same strike price K and expiry date T, computed by a Monte Carlo algorithm. Do $\widehat{C}$ and $\widehat{P}$ satisfy put-call parity? Justify your answer.

(d) If both estimators in part (c) are calculated using the same random sample $Z_1, \ldots, Z_M$ from a standard Normal distribution, find an expression for $\widehat{C} - \widehat{P}$.