Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4201-WE01

Title:

Analysis IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) Define what means for a set to be countable.

   (b) Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subset \mathbb{R}$. Show that $\mathbb{Q}[\sqrt{2}]$ is countable.

   (c) Show that the set of all infinite sequences of natural numbers is uncountable.

2. Let $\{a_n\}$ be a sequence of real numbers, and let A be the set of all elements of $\{a_n\}$.

   (a) Define $\limsup a_n$.

   (b) Show that $\sup A \geq \limsup a_n$.

   (c) Assume that $\limsup a_n \in A$. Does this imply that $\sup A \in A$?

3. (a) Define what it means for a set $E \subset \mathbb{R}$ to be measurable.

   (b) Show that if $E \subset \mathbb{R}$ is measurable and bounded and $\varepsilon > 0$, then there exists a finite collection $\{E_1, \ldots, E_n\}$ of mutually disjoint measurable sets such that $\bigcup_{i=1}^n E_i = E$ and $m(E_i) \leq \varepsilon$ for all $i = 1, \ldots, n$.

   (c) Is the assertion of (b) true if E is unbounded of finite measure?

4. (a) Define what means for $f : \mathbb{R} \to \mathbb{R}$ to be measurable.

   (b) Prove that $f(x) := \sin(3x)$ is measurable. You may use that open intervals are measurable.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable, $a \in \mathbb{R}$ and define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f(x - a)$. Prove that g is measurable.

5. (a) Define $\int f$ for a nonnegative measurable function $f : \mathbb{R} \to \mathbb{R}$.

   (b) Prove that for measurable functions f, g with $0 \leq f(x) \leq g(x)$ we have $\int f \leq \int g$.

   (c) Let f be as in (a), assume that $\int f = 0$, $c > 0$, and $A_c := \{x \in \mathbb{R} | f(x) > c\}$. Prove that $m(A_c) = 0$, where m denotes the Lebesgue measure of $\mathbb{R}$.

6. (a) State the Lemma of Fatou.

   (b) Let

   $$f_n(x) = \begin{cases} 0, & x \leq n; \\ 1, & x > n. \end{cases}$$

   Do the **assumptions** of the Lemma of Fatou apply to the sequence? Prove your answer.

   (c) Does the **conclusion** of the Lemma of Fatou apply to the sequence f_n as in (b)? Prove your answer.

SECTION B

7. (a) Define the outer measure $m^*(E)$ of a set $E \subseteq \mathbb{R}$.

   (b) Use the definition (a) to show that the outer measure is monotone, i.e. $m^*(A) \leq m^*(B)$ for $A \subseteq B$.

   (c) Show that a set $E \subseteq \mathbb{R}$ is measurable if and only if for every $\varepsilon > 0$ there exist an open set U and a closed set F such that $F \subseteq E \subseteq U$ and $m^*(U \setminus F) < \varepsilon$.

   (d) Let $E \subseteq \mathbb{R}$ have finite outer measure. Show that there exists a countable intersection G of open sets such that $E \subseteq G$ and $m^*(E) = m^*(G)$. Does this imply that $m^*(G \setminus E) = 0$?

8. A set $A \subseteq \mathbb{R}$ is called *nowhere dense* if every open non-empty set $U \subseteq \mathbb{R}$ has an open non-empty subset $U_0 \subseteq U$ such that $U_0 \cap A = \emptyset$. A set $B \subseteq \mathbb{R}$ is called *dense* if the closure of B coincides with $\mathbb{R}$.

   (a) Let $N \in \mathbb{N}$. Show that the set $\{\frac{p}{q} \,|\, p, q \in \mathbb{N},\, p \leq N\}$ is nowhere dense.

   (b) Let $B \subseteq \mathbb{R}$ be dense. Is it true that $\mathbb{R} \setminus B$ has to be nowhere dense?

   (c) Let $A \subseteq \mathbb{R}$ be nowhere dense. Is it true that $\mathbb{R} \setminus A$ has to be dense?

   (d) Show that for every $\varepsilon > 0$ there exists a nowhere dense set $E \subset \mathbb{R}$ such that $m^*(\mathbb{R} \setminus E) < \varepsilon$.

9. (a) Define the collection of functions $L^1([0, 2\pi])$ and the collection of functions $L^2([0, 2\pi])$.

   (b) Prove or disprove by counterexample the following claim : $L^1([0, 2\pi]) \subset L^2([0, 2\pi])$.

   (c) State the Dominated Convergence Theorem.

10. (a) Define an inner product on $L^2([0, 2\pi])$ and define what it means that $L^2([0, 2\pi])$ is a Hilbert space.

    (b) State and prove the Bessel Inequality on the Hilbert space $L^2([0, 2\pi])$.

    (c) Define functions $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = (1/n)\chi_{[0,n]}$ and $f(x) = 0$. Here, $\chi_{[0,n]}$ has value 1 on $[0, n]$ and vanishes elsewhere. Show that f_n converges uniformly to f but that $\lim \int f_n \neq \int f$. Why does this not contradict the Monotone Convergence Theorem ?

SECTION C

11. (a) State Egoroff's Theorem.

    (b) Let $f_n : [-\pi/2, \pi/2] \to \mathbb{R}$, $f_n(x) = \frac{\tan x}{n}$ for $x \in (-\pi/2, \pi/2) \cap (\mathbb{R} \setminus \mathbb{Q})$, $f_n(x) = 0$ for $x \in [-\pi/2, \pi/2] \cap \mathbb{Q}$. Show that all f_n are measurable.

    (c) Show that there exists a pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$. Find explicitly a closed set $F \subset [-\pi/2, \pi/2]$ such that $m([-\pi/2, \pi/2] \setminus F) < 1/5$ and the convergence of $\{f_n\}$ to f on F is uniform.

    (d) Give an example of a sequence of measurable functions $g_n : \mathbb{R} \to \mathbb{R}$ which converges pointwise to a function $g : \mathbb{R} \to \mathbb{R}$ and to which the **conclusion** of the Egoroff's Theorem does not apply.