

## **EXAMINATION PAPER**

Examination Session: May

2018

Year:

Exam Code:

MATH4201-WE01

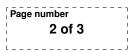
Title:

Analysis IV

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best <b>TWO</b> answers from Section the best <b>THREE</b> answers from Section <b>AND</b> the answer to the question in Section B and C carry <b>T</b> those in Section A.	on B, ection C.	any marks as

**Revision:** 



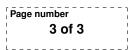
## SECTION A

- 1. (a) Define what means for a set to be countable.
  - (b) Let  $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subset \mathbb{R}$ . Show that  $\mathbb{Q}[\sqrt{2}]$  is countable.
  - (c) Show that the set of all infinite sequences of natural numbers is uncountable.
- 2. Let  $\{a_n\}$  be a sequence of real numbers, and let A be the set of all elements of  $\{a_n\}$ .
  - (a) Define  $\limsup a_n$ .
  - (b) Show that  $\sup A \ge \limsup a_n$ .
  - (c) Assume that  $\limsup a_n \in A$ . Does this imply that  $\sup A \in A$ ?
- 3. (a) Define what it means for a set  $E \subset \mathbb{R}$  to be measurable.
  - (b) Show that if  $E \subset \mathbb{R}$  is measurable and bounded and  $\varepsilon > 0$ , then there exists a finite collection  $\{E_1, \ldots, E_n\}$  of mutually disjoint measurable sets such that  $\bigcup_{i=1}^n E_i = E$  and  $m(E_i) \leq \varepsilon$  for all  $i = 1, \ldots, n$ .
  - (c) Is the assertion of (b) true if E is unbounded of finite measure?
- 4. (a) Define what means for  $f : \mathbb{R} \to \mathbb{R}$  to be measurable.
  - (b) Prove that  $f(x) := \sin(3x)$  is measurable. You may use that open intervals are measurable.
  - (c) Let  $f : \mathbb{R} \to \mathbb{R}$  be measurable,  $a \in \mathbb{R}$  and define the function  $g : \mathbb{R} \to \mathbb{R}$  by g(x) := f(x a). Prove that g is measurable.
- 5. (a) Define  $\int f$  for a nonnegative measurable function  $f : \mathbb{R} \to \mathbb{R}$ .
  - (b) Prove that for measurable functions f, g with  $0 \le f(x) \le g(x)$  we have  $\int f \le \int g$ .
  - (c) Let f be as in (a), assume that  $\int f = 0, c > 0$ , and  $A_c := \{x \in \mathbb{R} | f(x) > c\}$ . Prove that  $m(A_c) = 0$ , where m denotes the Lebesgue measure of  $\mathbb{R}$ .
- 6. (a) State the Lemma of Fatou.
  - (b) Let

$$f_n(x) = \begin{cases} 0, \ x \le n; \\ 1, \ x > n. \end{cases}$$

Do the **assumptions** of the Lemma of Fatou apply to the sequence? Prove your answer.

(c) Does the **conclusion** of the Lemma of Fatou apply to the sequence  $f_n$  as in (b)? Prove your answer.



## SECTION B

- 7. (a) Define the outer measure  $m^*(E)$  of a set  $E \subseteq \mathbb{R}$ .
  - (b) Use the definition (a) to show that the outer measure is monotone, i.e.  $m^*(A) \le m^*(B)$  for  $A \subseteq B$ .
  - (c) Show that a set  $E \subseteq \mathbb{R}$  is measurable if and only if for every  $\varepsilon > 0$  there exist an open set U and a closed set F such that  $F \subseteq E \subseteq U$  and  $m^*(U \setminus F) < \varepsilon$ .
  - (d) Let  $E \subseteq \mathbb{R}$  have finite outer measure. Show that there exists a countable intersection G of open sets such that  $E \subseteq G$  and  $m^*(E) = m^*(G)$ . Does this imply that  $m^*(G \setminus E) = 0$ ?
- 8. A set  $A \subseteq \mathbb{R}$  is called *nowhere dense* if every open non-empty set  $U \subseteq \mathbb{R}$  has an open non-empty subset  $U_0 \subseteq U$  such that  $U_0 \cap A = \emptyset$ . A set  $B \subseteq \mathbb{R}$  is called *dense* if the closure of B coincides with  $\mathbb{R}$ .
  - (a) Let  $N \in \mathbb{N}$ . Show that the set  $\{\frac{p}{q} \mid p, q \in \mathbb{N}, p \leq N\}$  is nowhere dense.
  - (b) Let  $B \subseteq \mathbb{R}$  be dense. Is it true that  $\mathbb{R} \setminus B$  has to be nowhere dense?
  - (c) Let  $A \subseteq \mathbb{R}$  be nowhere dense. Is it true that  $\mathbb{R} \setminus A$  has to be dense?
  - (d) Show that for every  $\varepsilon > 0$  there exists a nowhere dense set  $E \subset \mathbb{R}$  such that  $m^*(\mathbb{R} \setminus E) < \varepsilon$ .
- 9. (a) Define the collection of functions  $L^1([0, 2\pi])$  and the collection of functions  $L^2([0, 2\pi])$ .
  - (b) Prove or disprove by counterexample the following claim :  $L^1([0, 2\pi]) \subset L^2([0, 2\pi])$ .
  - (c) State the Dominated Convergence Theorem.
- 10. (a) Define an inner product on  $L^2([0, 2\pi])$  and define what it means that  $L^2([0, 2\pi])$  is a Hilbert space.
  - (b) State and prove the Bessel Inequality on the Hilbert space  $L^2([0, 2\pi])$ .
  - (c) Define functions  $f_n : \mathbb{R} \to \mathbb{R}$  by  $f_n(x) = (1/n)\chi_{[0,n]}$  and f(x) = 0. Here,  $\chi_{[0,n]}$  has value 1 on [0, n] and vanishes elsewhere. Show that  $f_n$  converges uniformly to f but that  $\lim \int f_n \neq \int f$ . Why does this not contradict the Monotone Convergence Theorem ?

## SECTION C

- 11. (a) State Egoroff's Theorem.
  - (b) Let  $f_n : [-\pi/2, \pi/2] \to \mathbb{R}$ ,  $f_n(x) = \frac{\tan x}{n}$  for  $x \in (-\pi/2, \pi/2) \cap (\mathbb{R} \setminus \mathbb{Q})$ ,  $f_n(x) = 0$  for  $x \in [-\pi/2, \pi/2] \cap \mathbb{Q}$ . Show that all  $f_n$  are measurable.
  - (c) Show that there exists a pointwise limit  $f(x) = \lim_{n\to\infty} f_n(x)$ . Find explicitly a closed set  $F \subset [-\pi/2, \pi/2]$  such that  $m([-\pi/2, \pi/2] \setminus F) < 1/5$  and the convergence of  $\{f_n\}$  to f on F is uniform.
  - (d) Give an example of a sequence of measurable functions  $g_n : \mathbb{R} \to \mathbb{R}$  which converges pointwise to a function  $g : \mathbb{R} \to \mathbb{R}$  and to which the **conclusion** of the Egoroff's Theorem does not apply.