Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4241-WE01

Title: Representation Theory IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Let G be a finite group.

   (a) Define the regular representation (π, V) of G and state the theorem about the decomposition of V into irreducible representations. What is $\dim \mathrm{Hom}_G(V, V)$, the dimension of the space of G-intertwiners of V with itself?

   (b) Assume G is abelian and fix $g \in G$. Show that $\pi(g)$ is a G-intertwiner.

   (c) Assume $G = \mathbb{Z}/3\mathbb{Z}$. Write down explicitly the matrix $\pi(g)$ for every $g \in G$ and explain why these give a basis for $\mathrm{Hom}_G(V, V)$.

2. (a) Let (ρ, V) be a representation of a finite group G. Define the dual representation (ρ^*, V^*). What is the dual representation in terms of matrices? Show that

$$\chi_{V^*}(g) = \overline{\chi_V(g)}.$$

   (b) Explain why every finite-dimensional representation of the dihedral group D_n is equivalent to its dual representation.

3. (a) Give the character table of the symmetric group S_3. (No justification required).

   (b) Let f be the class function on S_3 defined by $f(C) = \#C$ for a conjugacy class C in S_3. Express f in terms of the irreducible characters of S_3. Does f arise as the character of a representation of S_3? Justify your answer.

4. Let G be a linear Lie group and $\mathfrak{g}$ be its Lie algebra.

   (a) Give explicit formulas for the adjoint representations ad and Ad.

   (b) Show that ad is the derived representation of Ad.

   (c) Show directly that ad is a Lie algebra homomorphism.

5. Let $G = \mathrm{SL}_2(\mathbb{C})$ and consider the action of G on the space of smooth functions on *row* vectors $\mathbf{x} \in \mathbb{C}^2$ given by

$$(\pi(g)\varphi)(\mathbf{x}) = \varphi(\mathbf{x}g).$$

   (a) Show that π defines a representation, i.e., $\pi(gh) = \pi(g)\pi(h)$ for $g, h \in G$.

   (b) Compute the associated derived Lie algebra action $D\pi$ for $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$.

6. ***There is no question 6 on this paper.***

SECTION B

7. For $n \geq 2$, we let Dic_n be the so-called dicyclic group. This is by definition the group generated by two elements a and x subject to the relations

$$a^{2n} = e, \qquad x^2 = a^n, \qquad x^{-1}ax = a^{-1}.$$

You may assume that Dic_n is a non-abelian group of order $4n$ and that every element can be written uniquely in the form $a^i x^j$ with $0 \leq i < 2n$ and $j = 0, 1$.

(a) Show that every irreducible representation of Dic_n has degree at most 2.

(b) Use the 'method of the abelian subgroup' to explicitly construct all (equivalence classes of) irreducible representations (ρ, V) of Dic_n. (For convenience handle the cases $\dim V = 1$ and $\dim V = 2$ separately). Write down the matrices $\rho(x)$ and $\rho(a)$ explicitly. Check that your list is complete.

(c) For $n = 3$, write down the character table of Dic_n.
(Hint: we have $\omega + \omega^{-1} = 1$; $\omega^2 + \omega^{-2} = -1$ for $\omega = e^{\pi i/3}$. You may initially assume that a, a^2, a^3, x, ax lie in different conjugacy classes. After completing the table explain why this is indeed the case. Then find the order of these classes using the table.)

8. For an *odd* prime p, we consider the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Recall all its (additive) characters are given by $\psi_\ell(x) := e^{2\pi i x \ell/p}$ with $\ell \in \mathbb{F}_p$. (Here we identify elements in $\mathbb{F}_p$ with their preimage in $\mathbb{Z}$).

Recall that the Heisenberg group H over $\mathbb{F}_p$ is given as the set of triples $(x, y, z) \in \mathbb{F}_p^3$ with the multiplication defined by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy' - x'y).$$

For example, we have $(-x, 0, 0) \cdot (0, y, z) \cdot (x, 0, 0) = (0, y, z - 2xy)$ and that the center of H is $\{(0, 0, z); z \in \mathbb{F}_p\}$.

(a) State Mackey's irreduciblity criterion for induced representations from a *normal* subgroup.

(b) Let $K = \{(0, y, z);\ y, z \in \mathbb{F}_p\}$, which you may assume is a normal abelian subgroup of H of index p with left coset representatives $\{(x, 0, 0)\}$. Consider for $\ell \neq 0$ its one-dimensional representation ψ'_ℓ given by $\psi'_\ell((0, y, z)) := \psi_\ell(z)$. Use (a) to show that the induced representation $\mathrm{Ind}_K^H \psi'_\ell$ is irreducible.

(c) Let (ρ, V) be an arbitrary representation of H. Assume that the restriction $\mathrm{Res}_K V$ to K contains the representation ψ'_ℓ. Show $\langle \mathrm{Ind}_K^H \psi'_\ell, V \rangle_H \geq 1$.

(d) Now assume that (ρ, V) is irreducible with central character ψ_ℓ, i.e., $\rho(0, 0, z)v = \psi_\ell(z)v$ for all $v \in V$. Note that V must contain an eigenvector w under the action of K: More precisely, (you may assume) there exists a $k \in \mathbb{F}_p$ such that

$$\rho(0, y, z)w = \psi_k(y)\psi_\ell(z)w \qquad \text{for all } (0, y, z) \in K.$$

Compute the action of $(0, y, z)$ on $w_x := \rho(x, 0, 0)w$ and show that there exists an $x \in \mathbb{F}_p$ such that

$$\rho(0, y, z)w_x = \psi_\ell(z)w_x \qquad \text{for all } (0, y, z) \in K.$$

In particular, $\mathrm{Res}_K V$ contains ψ'_ℓ. Use (c) to conclude $\mathrm{Ind}_K^H \psi'_\ell \simeq V$.

9. We define elements of $\mathfrak{sl}_2(\mathbb{C})$,

$$\widetilde{H} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad X_+ := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad X_- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

(a) Show that $\widetilde{H}$, X_+, and X_- satisfy exactly the same bracket relations as the standard basis $\{H, X, Y\}$ of $\mathfrak{sl}_2(\mathbb{C})$.

Show by an explicit calculation that if in a representation (π, V) of $\mathfrak{sl}_2(\mathbb{C})$ a vector v is an eigenvector for $\widetilde{H}$ ("$\widetilde{H}$-weight vector") with eigenvalue ("$\widetilde{H}$-weight") λ, then $\pi(X_\pm)v$ is either zero or also an $\widetilde{H}$-weight vector with $\widetilde{H}$-weight $\lambda \pm 2$.

(b) Let $\mathcal{F} := \mathbb{C}[z]$ be the (infinite-dimensional) $\mathbb{C}$-vector space of polynomials in one variable. We define operators on $\mathcal{F}$ by

$$\omega(\widetilde{H}) := z\frac{d}{dz} + \frac{1}{2}, \qquad \omega(X_+) := \frac{1}{2}z^2, \qquad \omega(X_-) := -\frac{1}{2}\frac{d^2}{dz^2}.$$

(So e.g., $\omega(\widetilde{H})p(z) = zp'(z) + \frac{1}{2}p(z)$.) Show that these operators preserve the bracket relations and hence define a Lie algebra representation ω of $\mathfrak{sl}_2(\mathbb{C})$. (The operator identities $\frac{d}{dz}z = 1 + z\frac{d}{dz}$, $\frac{d^2}{dz^2}z^2 = 2 + 4z\frac{d}{dz} + z^2\frac{d^2}{dz^2}$ should help).

(c) Find two linear independent lowest weight vectors ($\omega(X_-)p = 0$) in $\mathcal{F}$ and their $\widetilde{H}$-weights. Show that these vectors each generate an (infinite dimensional) subrepresentation, say $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$.

In particular, describe $\mathcal{F}_1$ and $\mathcal{F}_2$ and give their $\widetilde{H}$-weight structure and the action of $X_\pm$.

Finally, show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are irreducible.

10. ***There is no question 10 on this paper.***

SECTION C

11. We let $\mathbb{F}_q$ be the finite field with q elements and let $G = \mathrm{GL}_2(\mathbb{F}_q)$. We define subgroups of G,

$$T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right);\, a, d \in \mathbb{F}_q^\times\}, \qquad N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right);\, b \in \mathbb{F}_q\}, \qquad B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right);\, a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}.$$

Note all characters, that is, one-dimensional representations of the abelian group T are given by $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi(d)$, where χ_1 and χ_2 are two (multiplicative) characters of $\mathbb{F}_q^\times$. We define the associated one-dimensional representation ρ_{χ_1,χ_2} of B by

$$\rho_{\chi_1,\chi_2}\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) = \chi_1(a)\chi_2(d).$$

Note that N acts trivially.

(a) Set $I(\chi_1, \chi_2) := \mathrm{Ind}_B^G \rho_{\chi_1,\chi_2}$. Use Mackey's irreducibility criterion to show in detail that $I(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1 \neq \chi_2$.

(b) Show that if $\chi_1 = \chi_2 =: \chi$, then $I(\chi, \chi)$ contains the one-dimensional representation of G given by $g \mapsto \chi(\det(g))$.

(c) Let (V, π) be an arbitrary representation of G and consider

$$V^N := \{v \in V; \pi(n)v = v \ \forall n \in N\},$$

the subspace of vectors fixed by N. Assume $V^N \neq 0$.
Show that V^N is a representation of T, that is, T preserves V^N. (The identity $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & bd/a \\ 0 & 1 \end{smallmatrix}\right)$ might be useful).
Conclude that V^N contains a vector v on which the abelian group T acts by $\pi(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right))v = \chi_1(a)\chi_2(d)v$ for some characters χ_1, χ_2 as above. Hence

$$\pi\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right)v = \chi_1(a)\chi_2(d)v.$$

(d) Assume (V, π) is an irreducible representation of G. Show that if V is contained in some $I(\chi_1, \chi_2)$, then $V^N \neq 0$. Conversely, if $V^N \neq 0$, then show V is contained in some $I(\chi_1, \chi_2)$.