Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1031-WE01

Title:
Discrete Mathematics

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **FOUR** answers. All questions carry the same marks.

Revision:	

ED01/2019

1. (a) Consider the letters CONCLUSION.
   - i. How many arrangements of these letters are there?
   - ii. How many arrangements contain the word LION as a contiguous string?
   - iii. How many arrangements contain neither of the words LION, CONSUL?

   (b) Consider the identity

$$\prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{n!\,2^n}, \quad (n \geq 1).$$

   - i. Prove the identity by induction.
   - ii. Give a combinatorial explanation of the identity. *Hint:* consider arranging people into pairs.

2. (a) i. State the inclusion-exclusion formula. Explain clearly the meaning of any special notation that you use.

   Consider 8-letter words constructed using the letters ABC.
   - ii. How many such words are there in total?
   - iii. How many contain at least one A and at least two Bs?

   (b) Peter buys 12 balls from a shop which has unlimited stocks of red, pink, black, white, and yellow balls (indistinguishable apart from colour). How many different collections of balls can he buy
   - i. in total?
   - ii. in which every colour is either used an even number of times or not at all?
   - iii. in which every colour is either used an odd number of times or not at all?

3. (a) For each of the following recurrence relations, valid for $n \geq 2$, determine a_n as a function of n, given the initial conditions $a_0 = 0$ and $a_1 = 1$.
   - i. $a_n = a_{n-1} + 6a_{n-2}$;
   - ii. $a_n = a_{n-1} + 6a_{n-2} - 6n + 7$.

   (b) In some competition, 10 players play in a round-robin singles format, so each player plays once against every other player. For each match, a winner gets 2 points; 0 points are awarded to a loser, or to both players in the event of a draw.
   - i. State the pigeon-hole principle.
   - ii. Suppose that no player won all of their matches in the competition. Show that two players in the league have the same points total.
   - iii. Suppose that there were 4 drawn matches in the competition. Show that one player accumulated at least 10 points.

4. (a) Anne is arranging her building blocks end-to-end in a line. The blocks are of two types: one of length 1 inch, and the other of length 4 inches. Assuming Anne has an unlimited supply of both types of block, let b_n denote the number of ways in which she can make a line of length n inches. Set $b_0 = 1$.

    i. Evaluate directly b_1, b_2, b_3 and b_4.
    ii. Write down a recurrence relation for b_n, $n \geq 4$.
    iii. Let $g(x)$ denote the generating function $\sum_{n=0}^{\infty} b_n x^n$. Use your recurrence relation from part (ii.) to derive an expression for $g(x)$ as the reciprocal of a polynomial.

   (b) Anne is choosing n of her toys to take on holiday. She can choose from (identical) action figures, (identical) building blocks, and colouring pencils. The pencils come in any of five colours, and are otherwise identical. She will take at least three action figures, at least five building blocks, and between one and eight pencils of each colour.

   Let c_n denote the number of ways in which she can select the n toys.

    i. Write down a generating function for c_n and express it as compactly as possible.
    ii. Use your generating function from part (i.) to find c_{29}.

5. (a) Let $G = (V, E)$ be a simple finite graph, with $|V| = n$.

    i. Show that if G is not connected then its complement must be connected.
    ii. Prove that if the minimum degree of each vertex is $\geq n/2$ then G is connected.
    iii. Find a graph which is isomorphic to its own complement.
    iv. Prove that a graph G with 22 vertices is not isomorphic to its own complement.

   (b) i. Give the definition of the n-cube Q_n.
    ii. Use the Handshaking Lemma to find the number of edges in Q_n.
    iii. For which values of n does Q_n have an Euler circuit? Explain.
    iv. A Hamilton cycle visits each vertex just once.
    Prove that, for all $n \geq 2$, Q_n has a Hamilton cycle.
    v. Show a Hamilton cycle in Q_3.

6. (a) Let $G = (V, E)$ be a planar graph.

   i. Write down Euler's formula which relates the numbers of vertices, edges, faces, and connected components for a planar graph.

   ii. Write down two "handshaking" lemmas.

   iii. The 4-cycles in G are the shortest cycles and $|E| = 50$, $|V| = 25$. Using this, and your results from (i.) and (ii.), prove by contradiction that G is not planar.

   (b) Let now $G = (V, E)$ be a simple planar graph.

   i. For the case $|V| \geq 3$, prove that $|E| \leq 3|V| - 6$. Show, by providing a counterexample, that this inequality may fail if $|V| < 3$.

   ii. Show that a complete graph K_5 is not planar.

   iii. Are the graphs K_n for $n \leq 4$ planar or not? Justify your answer.