Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1041-WE01

Title: Programming & Dynamics I

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (i) The position vector can be written as $\mathbf{r} = r\mathbf{e}_r$ in two-dimensional polar coördinates. Using the relations $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$, $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$, find expressions for the velocity and acceleration vectors in two-dimensional polar coördinates. Use these to find expressions in the $\mathbf{e}_r$, $\mathbf{e}_\theta$ basis for the velocity and acceleration of a particle whose trajectory is given by $r = t^2$, $\theta = t^{-1}$.

   (ii) A particle of unit mass moves in one dimension. The particle is subject to a force $F = -3v^2 + 5vx$ where its displacement is given by x and its velocity by v. Calculate v as a function of x, given that at $x = 0$ the particle's velocity is $v = 1$.

   (iii) Two particles move without friction on a horizontal surface. One particle has mass m and initially has velocity $\mathbf{v} = u\cos(\alpha)\mathbf{i} + u\sin(\alpha)\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are an orthonormal basis. The other particle has mass $2m$ and initially has velocity $-U\mathbf{i}$ where $U > 0$. The two particles collide and subsequently the particle of mass $2m$ is at rest. If no energy is lost in the collision, find U in terms of m, u and α.

2. (i) The displacement of a particle of unit mass moving in one dimension is given by x. The particle moves in a potential

$$V(x) = x^3 - 6x^2 + 18x - 18\ln(1 + x).$$

   (a) Find the points of equilibrium of the potential $V(x)$, stating which are stable and which are unstable.

   (b) Sketch the function $V(x)$.

   (c) Calculate the period of small oscillations around each of the stable equilibria you have found.

   (d) The particle is fired with speed u from the stable equilibrium point with higher potential energy towards the stable equilibrium point with lower potential energy. Show that to reach the other stable equilibrium point

$$u^2 > 36\ln\left(\frac{3}{2}\right) - 14.$$

   You may assume that $\ln(3) < 10/9$.

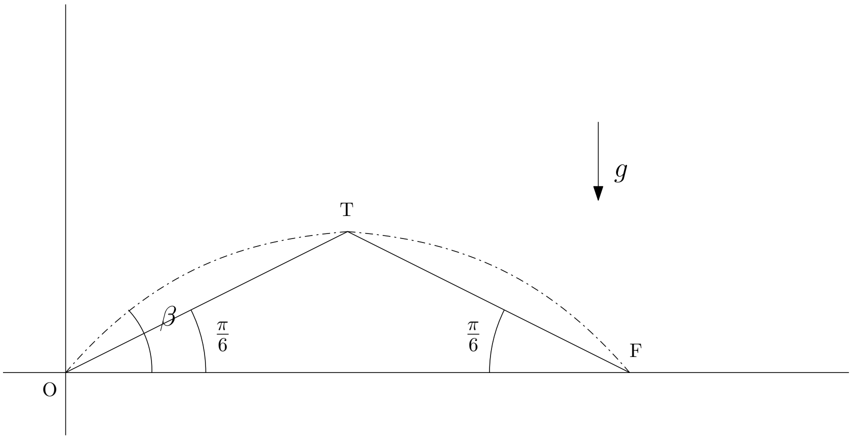
   (ii) A simple harmonic oscillator is subject to an additional force and is described by the equation

$$\ddot{z} + 9z = \sin(\omega t).$$

   (a) For what value of $\omega > 0$ does this equation describe a resonant system?

   (b) Given that ω takes this value find $z(t)$, if at $t = 0$, $z = 0$ and $\dot{z} = 1/2$.

3. The Pharaoh Thampthis IV constructs a pyramid whose sloping faces make an angle of $\pi/6$ to the horizontal as indicated in the diagram below. The dimensions



of the pyramid are chosen so that Pharoah's strongest warrior using all his might can throw a stone that just passes over the top of the pyramid T and lands at the base of the pyramid on the far side at F, as shown by the dotted line.

(a) If the warrior throws the stone from the origin O with speed U and at an angle β to the horizontal, find the stone's range and its maximum height, in terms of U, β and g, and deduce that if the stone passes through O, T and F then

$$\tan(\beta) = \frac{2}{\sqrt{3}}.$$

(b) Show that $\cos(\beta) = \sqrt{3/7}$ and $\sin(\beta) = \sqrt{4/7}$. It may help to use the relation $1 + \tan^2(\theta) = \sec^2(\theta)$.

(c) Find the coördinates of the top of the pyramid T in terms of U and g.

(d) Pharaoh wishes to fool the onlookers into thinking he is as strong as the warrior by throwing a stone from O so it lands out of sight just beyond the top of the pyramid T, hoping that the stone will then roll down the far side of the pyramid to F. Being an enlightened ruler he has attended a course on Dynamics, and therefore knows that the optimal angle to throw the stone at is $\pi/3$ to the horizontal, so that its initial velocity vector bisects the angle between the slope of the pyramid and the vertical. Show that if he throws it at this angle and the stone is to land on the other side he must throw it with speed u such that

$$u > \sqrt{\frac{6}{7}}U.$$

4. (i) A particle of unit mass and unit charge moves under the influence of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

   (a) Write down the Lorentz force which the particle experiences.

   (b) In a particular inertial frame of reference with basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, the electric and magnetic fields are both constant and explicitly given as $\mathbf{E} = 3\mathbf{i}$ and $\mathbf{B} = \mathbf{k}$ respectively. If the particle's velocity vector is $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, write down Newton's Second Law as three differential equations for the components v_1, v_2 and v_3. (You can ignore gravity in this question.)

   (c) Initially at $t = 0$ the particle is at rest at the origin. Use the differential equations in part (b) to find both $\mathbf{v}$ and the position of the particle at later times. Show that the particle passes through the points $(0, -6n\pi, 0)$ where n is an integer.

   (ii) The displacement of a string from its equilibrium position is described by a function $u(x,t)$. The string is fixed at both ends so that $u(0,t) = u(\pi,t) = 0$ for all t.

   (a) Write down the wave equation that $u(x,t)$ obeys, given that the speed of waves on the string is c.

   (b) Show that

$$u(x,t) = \sum_{n>0} \sin(nx)\left(A_n \cos(nct) + B_n \sin(nct)\right)$$

   where A_n and B_n are constants, is a solution to the wave equation which also satisfies the boundary conditions $u(0,t) = u(\pi,t) = 0$.

   (c) At $t = 0$, the string is at rest and is released from the position $u(x,0) = f(x)$ where

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

   Find the constants A_n and B_n and hence the displacement of the string $u(x,t)$ for $t > 0$.