Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1051-WE01

Title:
Analysis I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (a) State what it means for a real sequence $(x_n)_{n\in\mathbb{N}}$ to converge with limit $x^* \in \mathbb{R}$.
   (b) In each of the following cases evaluate $\lim_{n\to\infty} x_n$ or show that it does not exist. Refer to any results you use.
      (i) $x_n = \cos(n^3)/\sqrt{n^2+3n}$.
      (ii) $x_n = n\big(\sqrt{n+1} - \sqrt{n-1}\big)^2$.
      (iii) $x_n = \big((n+2)/(n+1)\big)^{2n}$.

2. (a) Prove the uniqueness of limits theorem; that is, prove that every convergent sequence has precisely one limit.
   (b) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence and suppose that $(|x_n|)$ is convergent. Prove or give a counterexample to each of the following statements:
      (i) If $\lim_{n\to\infty} |x_n| > 0$ then the sequence (x_n) is convergent.
      (ii) If $\lim_{n\to\infty} |x_n| = 0$ then the sequence (x_n) is convergent.

3. (a) State the completeness axiom of the real numbers.
   (b) Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined inductively by $a_1 = 0$ and $a_{n+1} = \sqrt{a_n + 6}$.
      (i) Prove that the sequence (a_n) is bounded above by 3.
      (ii) Prove that the sequence (a_n) is monotone increasing.
      (iii) Find the supremum of $A = \{a_n \,:\, n \in \mathbb{N}\}$.

4. (a) State the Bolzano-Weierstrass theorem.
   (b) Say what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to be Cauchy.
   (c) Show that every real, bounded Cauchy sequence converges with a real limit.

5. (a) Give the $(\varepsilon,\, \delta)$-definition of continuity of a function $f : X \longrightarrow \mathbb{R}$ at a point $x = c \in X$, where X is an open subset of $\mathbb{R}$.
   (b) Let $g : (0, \infty) \longrightarrow \mathbb{R}$ be given by $g(x) = 1/x^2$. Show, using the definition of continuity in (a), that g is continuous at $x = 1$.
   (c) Using quantifiers, give the negation of the statement in part (a); that is, give the precise logical statement that the function f is not continuous at $x = c$.
   (d) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$h(x) = \begin{cases} x^2 & \text{if } x > 2, \\ x+1 & \text{if } x \leq 2. \end{cases}$$

   Show that h is not continuous at $x = 2$.

6. (a) State the Mean Value Theorem.

   (b) Assume $f(x)$ is a differentiable function on a closed and bounded interval $[a, b]$ with $f'(x)$ continuous on $[a, b]$. Show carefully that there exists a constant $M > 0$ such that
$$|f(x) - f(y)| \leq M|x - y| \tag{1}$$
   for all x and y in $[a, b]$.

   (c) Consider $f(x) = \sqrt{x}$ on the interval $[0, 1]$. Show (by contradiction) that f does not satisfy (1). Why does this not contradict (b)?

7. (a) Find all $x \in \mathbb{R}$ such that $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}6^k}$ converges. When do we have (i) absolute and (ii) conditional convergence? State what results you use.

   (b) Let $\alpha > 0$. Compute $\displaystyle\lim_{x\to 0} \frac{\log(1 + x^\alpha)}{x^\alpha}$. Use this to determine whether the series $\displaystyle\sum_{k=1}^{\infty} \log(1 + \tfrac{1}{k^2})$ converges or diverges.

8. (a) Why is $\displaystyle\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$ an improper integral? Show carefully that this improper integral converges. [Hint: $\sqrt{1-t^4} = \sqrt{(1-t)(1+t+t^2+t^3)}$; do NOT try to evaluate the integral in any way.]

   (b) Show that $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx)}{k(k+1)}$ is an integrable function on the interval $[0, \pi/2]$ and evaluate $\displaystyle\int_0^{\pi/2} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k(k+1)}dx$ as an infinite series. Justify your steps. Then show that the value lies in the interval $(17/36, 17/36 + 1/150)$.

9. Define a function on $[-1, 1]$ by $f(x) := \displaystyle\int_0^x \frac{1}{\sqrt{1-t^4}}dt$. You may use that f is also defined at $x = 1$ by problem 8(a).

   (a) Set $\alpha = f(1)$. Why is f also defined at $x = -1$ with $f(-1) = -\alpha$?

   (b) Show that f is differentiable on $(-1, 1)$ and a strictly monotone increasing function mapping $[-1, 1]$ to $[-\alpha, \alpha]$.

   (c) Show that the inverse function $g(u)$ exists and is differentiable in $(-\alpha, \alpha)$ satisfying $g(u)^4 + [g'(u)]^2 = 1$. Conclude that $g'(u) \neq 0$ for all $u \in (-\alpha, \alpha)$.

   (d) Use (c) to show that $g''(u) = -2g(u)^3$ for all $u \in (-\alpha, \alpha)$.

10. (a) Carefully state the definitions of a step function and of a regulated function on a closed and bounded interval $[a, b]$.

    (b) Let f be a function on $[0, 1]$ satisfying the inequality (1) in Problem 6(b). Show directly from the definitions that f is a regulated function.

    (c) Explain why $f(x) = \sqrt{x}$ is a regulated function on $[0, 1]$. State explicitly what results from the lectures you are using.