

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH1051-WE01

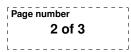
Title:

Analysis I

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.		
		Devision	

Revision:



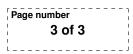
- 1. (a) State what it means for a real sequence $(x_n)_{n \in \mathbb{N}}$ to converge with limit $x^* \in \mathbb{R}$.
 - (b) In each of the following cases evaluate $\lim_{n\to\infty} x_n$ or show that it does not exist. Refer to any results you use.

(i)
$$x_n = \cos(n^3)/\sqrt{n^2 + 3n}$$
.
(ii) $x_n = n(\sqrt{n+1} - \sqrt{n-1})^2$

- (iii) $x_n = ((n+2)/(n+1))^{2n}$.
- 2. (a) Prove the uniqueness of limits theorem; that is, prove that every convergent sequence has precisely one limit.
 - (b) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence and suppose that $(|x_n|)$ is convergent. Prove or give a counterexample to each of the following statements:
 - (i) If $\lim_{n\to\infty} |x_n| > 0$ then the sequence (x_n) is convergent.
 - (ii) If $\lim_{n\to\infty} |x_n| = 0$ then the sequence (x_n) is convergent.
- 3. (a) State the completeness axiom of the real numbers.
 - (b) Let $(a_n)_{n \in \mathbb{N}}$ be the sequence defined inductively by $a_1 = 0$ and $a_{n+1} = \sqrt{a_n + 6}$.
 - (i) Prove that the sequence (a_n) is bounded above by 3.
 - (ii) Prove that the sequence (a_n) is monotone increasing.
 - (iii) Find the supremum of $A = \{a_n : n \in \mathbb{N}\}.$
- 4. (a) State the Bolzano-Weierstrass theorem.
 - (b) Say what it means for a sequence $(x_n)_{n \in \mathbb{N}}$ to be Cauchy.
 - (c) Show that every real, bounded Cauchy sequence converges with a real limit.
- 5. (a) Give the (ε, δ) -definition of continuity of a function $f : X \longrightarrow \mathbb{R}$ at a point $x = c \in X$, where X is an open subset of \mathbb{R} .
 - (b) Let $g: (0, \infty) \longrightarrow \mathbb{R}$ be given by $g(x) = 1/x^2$. Show, using the definition of continuity in (a), that g is continuous at x = 1.
 - (c) Using quantifiers, give the negation of the statement in part (a); that is, give the precise logical statement that the function f is not continuous at x = c.
 - (d) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$h(x) = \begin{cases} x^2 & \text{if } x > 2, \\ x+1 & \text{if } x \le 2. \end{cases}$$

Show that h is not continuous at x = 2.



- 6. (a) State the Mean Value Theorem.
 - (b) Assume f(x) is a differentiable function on a closed and bounded interval [a, b] with f'(x) continuous on [a, b]. Show carefully that there exists a constant M > 0 such that

$$|f(x) - f(y)| \le M|x - y| \tag{1}$$

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for all x and y in [a, b].

- (c) Consider $f(x) = \sqrt{x}$ on the interval [0, 1]. Show (by contradiction) that f does not satisfy (1). Why does this not contradict (b)?
- 7. (a) Find all $x \in \mathbb{R}$ such that $\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}6^k}$ converges. When do we have (i) absolute and (ii) conditional convergence? State what results you use.
 - (b) Let $\alpha > 0$. Compute $\lim_{x \to 0} \frac{\log(1 + x^{\alpha})}{x^{\alpha}}$. Use this to determine whether the series $\sum_{k=1}^{\infty} \log(1 + \frac{1}{k^2})$ converges or diverges.
- 8. (a) Why is $\int_0^1 \frac{1}{\sqrt{1-t^4}} dt$ an improper integral? Show carefully that this improper integral converges. [Hint: $\sqrt{1-t^4} = \sqrt{(1-t)(1+t+t^2+t^3)}$; do NOT try to evaluate the integral in any way.]
 - (b) Show that $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k(k+1)}$ is an integrable function on the interval $[0, \pi/2]$ and evaluate $\int_{0}^{\pi/2} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k(k+1)} dx$ as an infinite series. Justify your steps. Then show that the value lies in the interval (17/36, 17/36 + 1/150).
- 9. Define a function on [-1,1] by $f(x) := \int_0^x \frac{1}{\sqrt{1-t^4}} dt$. You may use that f is also defined at x = 1 by problem 8(a).
 - (a) Set $\alpha = f(1)$. Why is f also defined at x = -1 with $f(-1) = -\alpha$?
 - (b) Show that f is differentiable on (-1, 1) and a strictly monotone increasing function mapping [-1, 1] to $[-\alpha, \alpha]$.
 - (c) Show that the inverse function g(u) exists and is differentiable in $(-\alpha, \alpha)$ satisfying $g(u)^4 + [g'(u)]^2 = 1$. Conclude that $g'(u) \neq 0$ for all $u \in (-\alpha, \alpha)$.
 - (d) Use (c) to show that $g''(u) = -2g(u)^3$ for all $u \in (-\alpha, \alpha)$.
- 10. (a) Carefully state the definitions of a step function and of a regulated function on a closed and bounded interval [a, b].
 - (b) Let f be a function on [0, 1] satisfying the inequality (1) in Problem 6(b). Show directly from the definitions that f is a regulated function.
 - (c) Explain why $f(x) = \sqrt{x}$ is a regulated function on [0, 1]. State explicitly what results from the lectures you are using.