

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH1061-WE01

Title:

Calculus & Probability I paper 1: Calculus

Time Allowed:	1 hour 30 minutes			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.			

Revision:

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- 1. (a) Without using Taylor series or L'Hôpital's rule, calculate the limit

$$\lim_{x \to 3} \frac{x-3}{\sqrt{3x+7}-4}.$$

(b) Use L'Hôpital's rule to calculate the limit

$$\lim_{x \to \pi} \frac{\sin(3x)}{\sin(5x)}.$$

(c) Use Taylor series to calculate the limit

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2e^{2x^2}}{\sin(x^2)}.$$

- 2. (a) State Rolle's theorem.
 - (b) Let f(t) be differentiable for all $t \in \mathbb{R}$. For x > 0, define the function

$$g(t) = x(f(t) - f(0)) - t(f(x) - f(0)).$$

By applying Rolle's theorem to the function g(t) in the interval [0, x], prove that $\exists c \in (0, x)$ such that

$$f(x) = f(0) + xf'(c).$$

(c) Apply the result from part (b) to show that for all x > 0

$$e^{\sin^2 x} \le 1 + ex$$

3. Let y(x) be the solution of the initial value problem y(0) = 1 and

$$(2xy^3 - 3x^2y)dx + (3y^2(x^2 + 1) - x^3)dy = 0.$$

- (a) Show that this differential equation is exact.
- (b) Use the result from part (a) to find an implicit form for y(x).
- (c) Use the result from part (b) to show that y(1) = 1.
- (d) Use the result from part (c) to calculate y'(1).
- 4. (a) Derive the Jacobian for the change of variables from Cartesian coordinates $(x, y) \in \mathbb{R}^2$ to polar coordinates r, θ , where

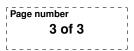
$$x = r\cos\theta, \qquad y = r\sin\theta.$$

(b) Let D be the region of the (x, y)-plane given by $x^2 + y^2 \le 1$. Use the change of variables given in part (a) to calculate the double integral

$$\iint_D \sqrt{1 - x^2 - y^2} \, dx \, dy.$$

(c) Explain why the result from part (b) can be used to conclude that the unit ball in \mathbb{R}^3 , given by $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$, has volume $4\pi/3$.

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5. The function f(x) has period 2π , that is $f(x + 2\pi) = f(x)$, and is given by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x \le 0\\ 3 & \text{if } 0 < x < \pi \end{cases}$$

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(a) Show that the Fourier series of f(x) has the form

$$\alpha + \beta \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin\left((2m-1)x\right)$$

and determine the values of the constants α and β .

(b) By evaluating the Fourier series at $x = \pi/2$, determine the value of

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1}.$$

(c) Apply Parseval's theorem to determine the value of

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}.$$