EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1061-WE01

Title: Calculus & Probability I paper 1: Calculus

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (a) Without using Taylor series or L'Hôpital's rule, calculate the limit

$$\lim_{x\to 3} \frac{x-3}{\sqrt{3x+7}-4}.$$

(b) Use L'Hôpital's rule to calculate the limit

$$\lim_{x\to \pi} \frac{\sin(3x)}{\sin(5x)}.$$

(c) Use Taylor series to calculate the limit

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2e^{2x^2}}{\sin(x^2)}.$$

2. (a) State Rolle's theorem.

(b) Let $f(t)$ be differentiable for all $t \in \mathbb{R}$. For $x > 0$, define the function

$$g(t) = x\big(f(t) - f(0)\big) - t\big(f(x) - f(0)\big).$$

By applying Rolle's theorem to the function $g(t)$ in the interval $[0, x]$, prove that $\exists\ c \in (0, x)$ such that

$$f(x) = f(0) + xf'(c).$$

(c) Apply the result from part (b) to show that for all $x > 0$

$$e^{\sin^2 x} \leq 1 + ex.$$

3. Let $y(x)$ be the solution of the initial value problem $y(0) = 1$ and

$$(2xy^3 - 3x^2y)dx + (3y^2(x^2+1) - x^3)dy = 0.$$

(a) Show that this differential equation is exact.

(b) Use the result from part (a) to find an implicit form for $y(x)$.

(c) Use the result from part (b) to show that $y(1) = 1$.

(d) Use the result from part (c) to calculate $y'(1)$.

4. (a) Derive the Jacobian for the change of variables from Cartesian coordinates $(x, y) \in \mathbb{R}^2$ to polar coordinates r, θ, where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

(b) Let D be the region of the (x, y)-plane given by $x^2 + y^2 \leq 1$.
Use the change of variables given in part (a) to calculate the double integral

$$\iint_D \sqrt{1 - x^2 - y^2}\, dxdy.$$

(c) Explain why the result from part (b) can be used to conclude that the unit ball in $\mathbb{R}^3$, given by $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$, has volume $4\pi/3$.

5. The function $f(x)$ has period 2π, that is $f(x+2\pi) = f(x)$, and is given by

$$f(x) = \begin{cases} -1 & \text{if } \; -\pi < x \leq 0 \\ 3 & \text{if } \quad 0 < x < \pi \end{cases}$$

(a) Show that the Fourier series of $f(x)$ has the form

$$\alpha + \beta \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin\big((2m-1)x\big)$$

and determine the values of the constants α and β.

(b) By evaluating the Fourier series at $x = \pi/2$, determine the value of

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1}.$$

(c) Apply Parseval's theorem to determine the value of

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}.$$