Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1061-WE02

Title:
Calculus & Probability I paper 2: Probability

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

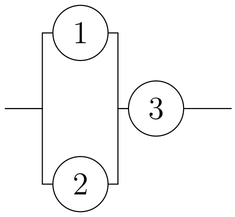
ED01/2019

1. (a) A bag contains 2 red marbles, 3 green marbles, and 5 blue marbles. Four marbles are randomly selected from the bag. Find the probability that the selection includes at least one marble of each colour.

   (b) Suppose that X is a continuous random variable with probability density function $f(x) = x$ for $0 < x < 1$, $f(x) = 2 - x$ for $1 \le x < 2$, and $f(x) = 0$ otherwise. Find the cumulative distribution function $F_X(t)$ of X, for all t.

   (c) Suppose that Y is a Poisson random variable with parameter $\lambda > 0$, so that $\mathbb{P}(Y = x) = \mathrm{e}^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \ldots$. Calculate $\mathbb{E}[Y(Y-1)]$.

2. A company manufactures Gizmos. The internal workings of each Gizmo can be represented by the following three-component reliability network:



Assume that after one year of operation, component i either works, with probability p_i, or not, with probability $1 - p_i$. Different components are independent.

   (a) Derive the probability $r(p_1, p_2, p_3)$ that a one-year old Gizmo works, as a function of p_1, p_2, p_3.

Components are manufactured by an unreliable machine. Every component that is produced is either *good* or *bad*. An assessment after one year of life shows that good components work with probability $3/4$ while bad components work with probability $1/2$. Assessment of the machine shows that each component manufactured is good with probability $2/3$, independently for each component.

   (b) Use this information to find p_1, p_2, p_3 and hence find the probability that a one-year old Gizmo works.

   (c) Suppose that a one-year old Gizmo fails. What is the probability that

      i. all three components were bad;

      ii. component 3 was bad but components 1 and 2 were good?

3. A fair coin is tossed n times. Let X be the total number of heads obtained, and let $R = X/n$ be the relative frequency of heads.

   (a) Compute $\mathbb{E}(R)$ and $\mathbb{V}\mathrm{ar}(R)$.

   (b) Use Markov's inequality to give an upper bound for $\mathbb{P}(R \geq 3/4)$.

   (c) Find the smallest value of n for which Chebyshev's inequality gives the bound
   $$\mathbb{P}\left(\left|R - \frac{1}{2}\right| \geq \frac{1}{20}\right) \leq \frac{1}{100}.$$

   (d) Find
   $$\lim_{n\to\infty} \mathbb{P}\left(\left|R - \frac{1}{2}\right| \geq \frac{1}{\sqrt{n}}\right).$$

   You may use the following values for the cumulative distribution function of the standard normal distribution:

x	0	1	2	3
$\Phi(x)$	0.5	0.841	0.977	0.999

4. A bag contains 1 red counter, 2 blue counters, and 3 green counters. Two counters are removed at random from the bag. Let X be the number of red counters removed, and let Y be the number of green counters removed.

   (a) Write down in a table the joint distribution of X and Y.

   (b) Find the marginal distributions of X and Y, and compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

   (c) Find $\mathbb{C}\mathrm{ov}(X, Y)$.

   (d) Find $\mathbb{E}(X \mid Y = y)$ for each possible value y of Y, and verify that, in this example, $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))$.

5. In this question, Γ denotes the Gamma function. In your answer you may use without proof the following identity (which you may take as the definition of Γ):
   $$\int_0^\infty x^{a-1}\mathrm{e}^{-bx}\mathrm{d}x = \frac{\Gamma(a)}{b^a}, \text{ for } a > 0,\ b > 0.$$

   For a positive integer k, a random variable X is said to have the $\chi^2(k)$ distribution, written $X \sim \chi^2(k)$, if it has probability density function
   $$f_k(x) = \begin{cases} \frac{x^{(k/2)-1}\mathrm{e}^{-x/2}}{2^{k/2}\Gamma(k/2)} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that the moment generating function of $X \sim \chi^2(k)$ is
   $$M_X(t) = (1 - 2t)^{-k/2} \text{ for } t < 1/2.$$

   (b) Suppose that $X \sim \chi^2(k)$ and $Y \sim \chi^2(\ell)$ are independent. What is the distribution of $X + Y$?

   (c) Show that if $X_k \sim \chi^2(k)$ for each k, then X_k/k converges in an appropriate sense as $k \to \infty$. Identify the limit.