Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1071-WE01

Title:
Linear Algebra I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Use a separate answer book for each Section.

Revision:	

ED01/2019

SECTION A

Use a separate answer book for this Section.

1. (a) Let $\mathbb{R}[x]_n$ be the set of polynomials of degree at most n. Assume $n \geq 1$. Show that
$$p(x) \mapsto \frac{d}{dx}p(x)$$
is a linear map $\mathbb{R}[x]_n \to \mathbb{R}[x]_{n-1}$.

   (b) Let $n \geq 3$ and consider the map $T : \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ given by
$$T : p(x) \mapsto 6p(x) + p'(x) - x^2 p''(x).$$
You may assume T is linear. Determine the nullity of T.

2. We write $M_n(\mathbb{R})$ for the set of $n \times n$ matrices with real entries.

   (a) Let $A \in M_n(\mathbb{R})$. Define
$$S_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$
by $S_A(B) = AB$ for all $B \in M_n(\mathbb{R})$. Show that S_A is a linear map.

   (b) Compute the rank and nullity of S_A when
$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

3. (a) Determine if the following set is a basis of $\mathbb{R}^4$.
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}.$$

   (b) For each $a \in \mathbb{R}$, determine those values of $X \in \mathbb{R}$ for which the determinant of the following matrix is 0.
$$\begin{pmatrix} a & X & a & a & a & a & a & a \\ X & a & a & a & a & a & a & a \\ a & a & a & X & a & a & a & a \\ a & a & X & a & a & a & a & a \\ a & a & a & a & a & X & a & a \\ a & a & a & a & X & a & a & a \\ a & a & a & a & a & a & a & X \\ a & a & a & a & a & a & X & a \end{pmatrix}.$$

4. (a) Determine if the following matrix is invertible and, if it is, compute the inverse matrix.
$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 5 \end{pmatrix}.$$

(b) Give the solution set to the following system of linear equations.

$$\begin{aligned} w + 4x + 3y + z &= 0, \\ w + 5x + 4y + 2z &= 0, \\ w - x + y - z &= 0. \end{aligned}$$

5. We write $M_n(\mathbb{R})$ for the set of $n \times n$ matrices with real entries.

For $A \in M_n(\mathbb{R})$, we define

$$U_A = \{B \in M_n(\mathbb{R}) : AB = BA\} \subseteq M_n(\mathbb{R})$$

and

$$V_A = \{B \in M_n(\mathbb{R}) : AB = -BA\} \subseteq M_n(\mathbb{R}).$$

You may assume that U_A and V_A are vector subspaces of $M_n(\mathbb{R})$.

(a) In the case

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

compute the dimensions of U_A, of V_A, and of $U_A \cap V_A$.

(b) Now let $A \in M_n(\mathbb{R})$. Suppose A has rank r for some r with $1 \leq r \leq n-1$. Show that $U_A \neq M_n(\mathbb{R})$.

SECTION B

Use a separate answer book for this Section.

6. Find the general solution to the system of first order differential equations

$$\begin{aligned}\dot{x}_1(t) &= 5x_1(t) + 2x_2(t) + x_3(t)\,,\\ \dot{x}_2(t) &= -8x_1(t) - 3x_2(t) - 2x_3(t)\,,\\ \dot{x}_3(t) &= 4x_1(t) + 2x_2(t) + 2x_3(t)\,.\end{aligned}$$

7. Let V be the vector space $\mathbb{R}[x]_2$ of real polynomials of degree at most two and let $\mathcal{L} : V \mapsto V$ be the linear operator

$$\mathcal{L}\,(p(x)) = \lambda x^2 p''(x) + xp'(x) + p(x+1)$$

with $p(x) \in \mathbb{R}[x]_2$, $p'(x) = dp(x)/dx$, and $\lambda \in \mathbb{R}$. Determine for which value of $\lambda \in \mathbb{R}$ the linear operator $\mathcal{L}$ has an eigenvalue equal to 0 and in that case find the corresponding eigenfunction.

8. (a) Show that

$$(\mathbf{z}, \mathbf{w}) = 3z_1\bar{w}_1 + iz_2\bar{w}_1 - iz_1\bar{w}_2 + 4z_2\bar{w}_2$$

defines an inner product on $V = \mathbb{C}^2$ and, using this inner product, find the norm of the vector

$$\mathbf{u} = \begin{pmatrix} i \\ i \end{pmatrix}\,.$$

(b) Let V be the vector space $\mathbb{R}[x]_2$ of real polynomials of degree at most two, with inner product

$$(p, q) = \int_0^1 p(x)q(x)\,dx\,.$$

Using this inner product find a basis for the orthogonal complement $U^\perp$ of the vector subspace $U = \mathrm{span}\{x\} \subseteq V$.

9. (a) Let A, B be $n \times n$ matrices such that $AB = BA$. If $\mathbf{v}$ is an eigenvector of A and if $B\mathbf{v} \neq \mathbf{0}$ show that $B\mathbf{v}$ is also an eigenvector for A.

(b) Let C be an $n \times n$ anti-hermitian matrix. Show that its eigenvalues are imaginary numbers, i.e. $\lambda = ix$ with $x \in \mathbb{R}$.

10. Let

$$G = \mathbb{Z}_2 \times \mathbb{Z}_3$$

be the direct product of the two cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Firstly write down the group table for G and secondly find an element of G with order 6.