Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1551-WE01

Title:
Mathematics for Engineers and Scientists

Time Allowed:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (i) Find the modulus and argument of $(i - \sqrt{3})^n$ for any $n \in \mathbb{Z}$.
(*You do not need to find the principal value of the argument.*)

(ii) Find all complex solutions of the equation $z^8 - 5iz^4 + 6 = 0$, leaving your answers in polar form.

(iii) State de Moivre's Theorem and use it to express $\cos(4\theta)$ as a function of $\cos\theta$. Deduce from your result that $\cos(\pi/8) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$.

2. (i) Compute the limit

$$\lim_{x \to 0} \frac{\sin^2(2x)}{\sin(3x)\sin(4x)}$$

carefully stating any standard results you use.

(ii) Use l'Hôpital's rule to compute the limit

$$\lim_{x \to 0} \left(\frac{1}{x \tan x} - \frac{1}{x^2} \right).$$

(iii) State the definition of the derivative of a function $f(x)$ as a limit. Use it to find the derivative of $1/x^3$ for $x \neq 0$.

3. (i) State Leibniz' Rule for the n^{th} derivative of a product $f(x) = u(x)v(x)$ and use it to find the 4^{th} derivative of $\dfrac{\ln x}{x^2}$.

(ii) Calculate the degree 2 Taylor polynomial of the function $f(x) = \sqrt{2 + x}$ about the point $x = 0$. Estimate the maximum error in using this polynomial to approximate $f(x)$ over the interval $0 \leq x \leq 0.1$.

(iii) Show that there is at least one solution to the equation $x^4 = 4x + 4$ in the interval $1 < x < 2$, stating any standard theorems you use.

Given an approximate solution x_n to the equation, find an iteration that should give a better approximation x_{n+1} and apply it to find x_1, starting from $x_0 = 2$.

4. (i) A plane passes through the points $A = (1, 2, 3)$, $B = (2, 3, 5)$ and $C = (1, 4, 6)$. Find the angle between a normal vector to the plane and the z-axis.

(ii) A strip is removed from the plane $x - 2y + 3z = 4$ by cutting along the planes $z = 0$ and $z = 1$. Find parametric equations for the edges of the strip and hence find its width.

(iii) Let $\mathbf{A}(x, y, z) = (A_1(x, y, z), A_2(x, y, z), A_3(x, y, z))$ be a vector field. Assuming that the order of mixed partial derivatives doesn't matter, prove that the divergence of the curl of $\mathbf{A}$ is zero, i.e. show that

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

5. (i) Let S be the surface with equation $x^2+2yz=2$ and let $\mathbf{p}$ be the point $(2,1,-1)$ on S. Find a Cartesian equation for the tangent plane to S at $\mathbf{p}$. Also find a Cartesian equation for the line L which is normal to S at $\mathbf{p}$ and hence find the other point where L intersects S.

   (ii) Determine all critical points of the function

$$f(x,y)=x^2y^2-x^2-y^2+1$$

   and identify each as a local minimum, local maximum or saddle point.

6. (i) Find the solution to the first order differential equation

$$y'-4xy=e^{2x^2}$$

   with initial condition $y(0)=1$.

   (ii) Find the solution to the second order differential equation

$$y''-4y=e^{2x}$$

   with initial conditions $y(0)=y'(0)=0$.

7. (i) Find the LU decomposition of the matrix

$$A=\begin{pmatrix}1 & 2 & 4\\ 3 & 8 & 14\\ -2 & 4 & -5\end{pmatrix}.$$

   Use it, together with forward and backward substitution to solve

$$\begin{aligned}x+2y+4z&=1\\ 3x+8y+14z&=3\\ -2x+4y-5z&=3.\end{aligned}$$

   (ii) Rearrange the system of equations

$$\begin{aligned}x+y-3z&=2\\ 2x-4y+z&=3\\ 4x-y-z&=1\end{aligned}$$

   so that it will converge under the Gauss-Seidel iteration method for any given initial values, stating why. Write out the resulting iteration and hence find the next approximation given that the initial guess is $\left(x^{(0)},y^{(0)}z^{(0)}\right)=(1,-1,-2)$.

8. Calculate the eigenvalues and corresponding eigenspaces of the matrix

$$A=\begin{pmatrix}3 & -1 & 2\\ 0 & 2 & 0\\ -1 & 1 & 0\end{pmatrix}.$$

Hence find an invertible matrix M and diagonal matrix D so that $M^{-1}AM=D$.

UNIVERSITY OF DURHAM

Formula sheet for **Mathematics for Engineers & Scientists (MATH1551/WE01)**

TRIGONOMETRIC FUNCTIONS

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\cos^2 A + \sin^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

$$\cos A\cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\sin A\sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

HYPERBOLIC FUNCTIONS

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh^{-1} x = \ln\left(x \pm \sqrt{x^2 - 1}\right)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh(iA) = \cos A$$

$$\sinh(iA) = i\sin A$$

$$\cosh^2 A - \sinh^2 A = 1$$

$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$$

$$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$$

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A\tanh B}$$

ELEMENTARY RULES FOR DIFFERENTIATION AND INTEGRATION

$$(u+v)' = u'+v', \quad (uv)' = u'v+uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \quad (u(v))' = u'(v)v', \quad \int u'v\,dx = uv - \int uv'\,dx$$

TAYLOR'S THEOREM

Taylor approximation: $f(x) \approx p_{n,a}(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$

and if $|f^{(n+1)}(x)| \leq M$ for $c \leq x \leq b$ then $|f(x) - p_{n,a}(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!}M$

DIFFERENTIAL OPERATORS

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (A_1, A_2, A_3) = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

TABLE OF DERIVATIVES

$y(x)$	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\ln x$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$

TABLE OF INTEGRALS

$f(x)$	$\int f(x)\,dx$
x^n	$\dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$
x^{-1}	$\ln \lvert x \rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln \lvert \cos x \rvert$
$\operatorname{cosec} x$	$-\ln \lvert \operatorname{cosec} x + \cot x \rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$
$\cot x$	$\ln \lvert \sin x \rvert$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \lvert \cosh x \rvert$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) \quad (a > x)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right) \quad (x > a)$

CRITICAL POINTS

Local maximum: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$

Local minimum: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$

Saddle point: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 < 0$

Inconclusive: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 = 0$

ITERATION METHODS

$$A\,\mathbf{x} = \mathbf{b}, \qquad A = D - L - U, \qquad T_j = D^{-1}(L+U), \qquad T_g = (D-L)^{-1}U$$

Jacobi's Method:

$$D\,\mathbf{x}^{(k+1)} = \mathbf{b} + (L+U)\,\mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = D^{-1}\,\mathbf{b} + D^{-1}(L+U)\,\mathbf{x}^{(k)}$$

Gauss-Seidel Method:

$$D\,\mathbf{x}^{(k+1)} = \mathbf{b} + L\,\mathbf{x}^{(k+1)} + U\,\mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = (D-L)^{-1}\,\mathbf{b} + (D-L)^{-1}U\,\mathbf{x}^{(k)}$$

SOR Method:

$$\mathbf{x}^{(k+1)} = (1-\omega)\,\mathbf{x}^{(k)} + \omega D^{-1}\left(\mathbf{b} + L\,\mathbf{x}^{(k+1)} + U\,\mathbf{x}^{(k)}\right)$$

Optimal Value: $$\omega = \frac{2}{1 + \sqrt{1 - \rho\,(T_j)^2}}$$