Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1561-WE01

Title: Single Mathematics A

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (a) Differentiate $x^{\exp(x)}$ with respect to x.

   (b) Find all real solutions x to $\tanh(\ln(x)) = 1/2$.

   (c) Compute the following limits without using L'Hôpital's rule

$$\lim_{x\to\infty} \sin(x^2)e^{-x} \qquad \lim_{x\to 3} \frac{(x-3)(x+8)}{2x^2-5x-3}$$

   State clearly which theorems you have used here, if any.

   (d) State the mathematical definition of continuity, *i.e.* what does it mean when we say that a function $f(x)$ is continuous at a point $x = a$?

   (e) Let

$$f(x) = \begin{cases} \dfrac{x\ln(x)}{\sin(\pi x)} & \text{for } 0 < x < 1 \text{ and } 1 < x < 2 \\ c & \text{for } x = 1 \end{cases}$$

   Determine c such that $f(x)$ is continuous at $x = 1$.

2. (a) Compute the following definite integral

$$\int_0^{\pi/6} \sin(x)\cos(2x)dx$$

   (b) Compute the following indefinite integral

$$\int \frac{x^2+x-4}{(x-7)(x^2+3)}dx$$

   (c) Compute the following indefinite integral

$$\int \frac{x^3}{\sqrt{1-x^2}}dx$$

   (d) Compute $|w|$, $\arg(w)$ and $\mathrm{Im}(w)$ for

$$w = \exp\left((\sqrt{3}+2i)e^{i\pi/3}\right).$$

3. (a) If $z = x + iy$ find coefficients a, b and c such that

$$|\sin(z)|^2 = a\cos^2(x) + b\sinh^2(y) + c\cosh^2(y)$$

   (b) Write

$$\frac{\sin(4\theta)}{\cos(\theta)}$$

   as a polynomial in $\sin(\theta)$.

   (c) Find all solutions z such that $\sinh(z) = 3\cosh(z)$.

   (d) Find all solutions z such that $4z^5 + 8z^3 + 5z = 0$. Clearly state the number of solutions you have found.

4. (a) Explain what it means for a series to converge absolutely and to converge conditionally.

   (b) Determine whether the series $\sum_{n=0}^{\infty} ne^{-n^2}$ converges or diverges.

   (c) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{n}\right)$ converges absolutely, converges conditionally or diverges.

   (d) Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[7]{n}}$ for $x \in \mathbb{R}$.

5. (a) Give the 3^{rd} Taylor polynomial $p_3(x)$ and the corresponding Lagrange form of the remainder for the function $f(x) = \sin(x)$ around the point $x = 0$.

   (b) Give the Taylor series around the point $x = 0$ for the function $\ln(1 + x)$.

   (c) Using the result in part(b), or otherwise, show that

$$(1+x)^{\frac{1}{x}} = e \cdot \left(1 - \frac{x}{2} + \frac{11x^2}{24}\right) + o(x^2)$$

   as $x \to 0$.

   (d) Find the 3^{rd} Taylor polynomial $p_3(x)$ of the function $f(x) = \int_0^x (1+t)^{1/t} dt$ around the point $x = 0$.

6. (a) Calculate the determinant of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 3 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

   (b) For which values of $\lambda \in \mathbb{R}$ does the following system of linear equations have one solution, infinitely many solutions, or no solutions?

$$\begin{cases} x + \lambda^2 y + z = \lambda, \\ \lambda x + y + (\lambda - 2)z = \lambda, \\ x + y + z = 1. \end{cases}$$

7. (a) Let $A = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}$. Determine the eigenvalues and eigenspaces of A.

   (b) Find the inverse of the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix}.$$

   (c) Write the matrix $A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 3 & 1 \\ 2 & 5 & 3 \end{pmatrix}$ as the sum of a symmetric and an anti-symmetric matrix.