Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH1571-WE01

Title: Single Mathematics B

Time Allowed:	3 hours	
Additional Material provided:	Tables: Normal distribution.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2019

1. (a) A particle of constant mass m has position vector

$$\boldsymbol{r}(t) = 3\cos(2t)\boldsymbol{i} + 3\sin(2t)\boldsymbol{j} - \cos(t)\boldsymbol{k}$$

at time t.

Calculate the angular momentum of the particle (about the origin), and the force acting on the particle.

(b) Find a Cartesian equation for the plane with normal vector $\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ containing the point with Cartesian coordinates $(3, 17, 1)$, and find the minimum distance from this plane to the origin.

(c) Sketch the two-dimensional polar basis vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ at a point with Cartesian coordinates $x > 0$ and $y > 0$, and use this to write an expression for $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ in terms of the Cartesian basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ with any coordinate dependence given in terms of the polar coordinates (r, θ).

Use your expressions to derive the result

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta$$

and derive a similar results for $\dot{\boldsymbol{e}}_\theta$ assuming the position is a function of time.

2. A particle moves in two dimensions with Cartesian coordinates $x(t)$ and $y(t)$ satisfying the following ordinary differential equations:

$$\begin{aligned} \dot{x} &= -x - \frac{9}{2}y \,, \\ \dot{y} &= 2x - y \,. \end{aligned}$$

At time $t = 0$ the position of the particle is $(0, 2)$.

(a) Use the above equations to find $\dot{x}(0)$ and $\dot{y}(0)$

(b) By differentiating the second equation and substituting to eliminate x, show that

$$\ddot{y} + 2\dot{y} + 10y = 0 \,.$$

(c) Solve the ODE in part (b) for $y(t)$, finding the unique solution satisfying the initial conditions.

(d) Using your solution for $y(t)$ (or by any other method), find $x(t)$.

3. A function of period 2π is defined by

$$f(x) = \begin{cases} -x^2 - 1 & , \quad -\pi < x < 0 \\ 0 & , \quad x = 0 \text{ or } x = \pi \\ x^2 + 1 & , \quad 0 < x < \pi \end{cases} .$$

(a) Sketch this function on the interval $(-3\pi, 3\pi)$ and state whether the function is odd, even or neither.

(b) Find the Fourier series for $f(x)$ on the interval $(-\pi, \pi)$.

(c) A function of period 2π is defined by

$$g(x) = \begin{cases} -x^3 - 3x & , \quad -\pi < x < 0 \\ 0 & , \quad x = 0 \text{ or } x = \pi \\ x^3 + 3x & , \quad 0 < x < \pi \end{cases} .$$

Find the Fourier Series for $g(x)$ by integrating the Fourier Series for $f(x)$ and write an expression for the integration constant by evaluating this Fourier Series at $x = 0$.

Alternatively, one could calculate the Fourier Series for $g(x)$ directly without using the Fourier Series for $f(x)$. Use this fact to find the exact value of the integration constant above.

4. (a) The potential at position $\boldsymbol{r}$ near a point dipole at the origin is $\phi = \boldsymbol{p} \cdot \boldsymbol{r}/r^3$ where $\boldsymbol{p}$ is a constant vector. Determine the force $\boldsymbol{F} = -\boldsymbol{\nabla}\phi$. Does $\boldsymbol{F}$ act centrally?

(b) Find and classify the critical points of the function

$$f(x, y) = \left(x^2 + y^2\right) e^{xy}.$$

(c) A thin cylindrical cup is made from clay of fixed area A. Determine what its radius r and height h should be in terms of A in order to maximize the volume V of liquid it holds. Find the maximum possible volume in terms of A.

5. (a) Evaluate the integral

$$\iint_A (y^2/x^2)\, dx dy$$

where A is the interior of the triangle with vertices $(0, 0), (1, 0)$ and $(1, 2)$.

(b) Evaluate the integral

$$\iiint_V (x^2 + y^2) dV$$

where V is the region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 3$.

6. (a) State the Cauchy-Riemann equations for a function $f = u(x, y) + iv(x, y)$. Show that the function $f(z) = \sin z$ (where $z = x + iy$) satisfies them.

   (b) A complex function is given by the following power series in $z = x + iy$:

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots + \frac{z^n}{2^n} + \ldots$$

   i. Using either the root formula test or the ratio test, determine a condition on z that is sufficient for the series to be convergent.

   ii. With reference to Taylor series, write an expression for $f(z)$ outside this radius of convergence.

   (c) Consider the vector field

$$\boldsymbol{V} \equiv (V_1,\, V_2,\, V_3) = \left(2x\log(yz)\,,\ \frac{x^2+y^2}{y} + 2y\log(yz)\,,\ \frac{x^2+z^2}{z} + 2z\log(yz)\right)$$

   and the associated differential $dF = V_1\,dx + V_2\,dy + V_3\,dz$.

   i. Determine if dF is exact.

   ii. Hence or otherwise find $\boldsymbol{\nabla} \times \boldsymbol{V}$ and $\boldsymbol{\nabla} \times (y\boldsymbol{V})$.

7. (a) A new species of fly is found to be a carrier of a tropical disease. Experimental studies on a sample of 1000 flies have determined that 75% of males in the sample are carriers of the disease, whereas only 60% of the females are carriers of the disease. Suppose that the sample consists of 400 males and 600 females.

   i. What is the probability that a fly selected at random from the sample is a carrier of the disease?

   ii. A sequence of trials is performed where a fly is selected at random from the sample and it is recorded whether the fly is a carrier of the disease. What is the probability that in three trials at least two carriers were selected?

   iii. Using an appropriate approximation, estimate the probability that in 100 trials at least 60 carriers were selected.

   iv. What is the (conditional) probability that a fly selected at random from the sample is female, given that it is a carrier of the disease?

   (b) Suppose X is an Exponential random variable with rate $\lambda > 0$, so that X has probability density function

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

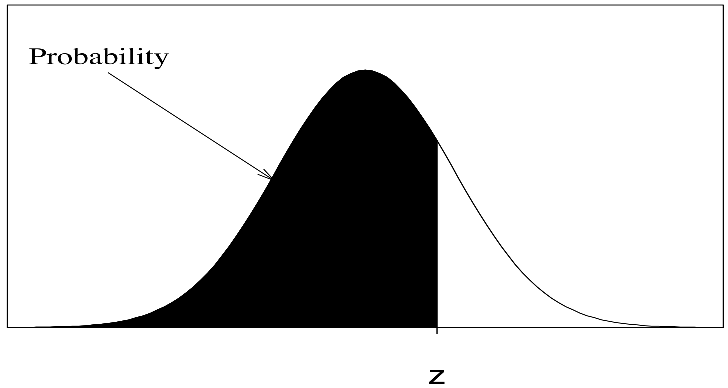
   i. Calculate $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

   ii. Let $X_1, X_2, \ldots, X_{100}$ be a sample of independent identically distributed copies of an Exponential random variable with rate 1. Calculate $\mathbb{E}(\bar{X})$ and $\mathbb{V}\mathrm{ar}(\bar{X})$, where $\bar{X}$ is the sample mean.

   iii. Use the Central Limit Theorem to find an approximate value for $\mathbb{P}(\bar{X} > 1.234)$.

Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998