EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2011-WE01

Title:
Complex Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let $U \subset \mathbb{C}$ be an open set. Define what it means for a function $f : U \to \mathbb{C}$ to be complex differentiable at a point $z_0 \in U$.

   (b) State the Cauchy-Riemann equations.

   (c) Let $U = \{z = x + iy \in \mathbb{C} \ : \ x > 0\}$ be the right half plane. Consider the function $f : U \to \mathbb{C}$ defined by

$$f(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\arctan\left(\frac{y}{x}\right).$$

   Here log denotes the real logarithm. Use the Cauchy-Riemann equations to determine the points $z_0 \in U$ where f is complex differentiable.

2. (a) Describe all the Möbius transformations that map the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$ to itself.

   (b) Find a Möbius transformation that maps $\mathbb{H}$ to itself and maps i to i and $-1+i$ to $1+2i$.

3. (a) Let $U \subset \mathbb{C}$, and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : U \to \mathbb{C}$. State the M-test for the series $\sum_{n=1}^{\infty} f_n$ to converge uniformly on U.

   (b) Prove that for any r with $0 < r < 1$ the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$$

   converges uniformly on $\{z \ : \ |z| \leq r\} \subset \mathbb{C}$.

4. (a) Find all the zeros and poles, with their orders, of

$$f(z) = \frac{z}{\sin z + \cos z}.$$

   (b) Find the residue of f at each of its poles.

5. (a) State Liouville's theorem.

   (b) Let f be a holomorphic function on $\mathbb{C} - \{0\}$. Show that f is bounded if and only if f is constant. State clearly any results you use from lectures.

6. (a) State Cauchy's Theorem for starlike domains.

   (b) Let

$$f(z) = \frac{1}{z^2} + e^{z^2}$$

   be defined on $\mathbb{C} - \{0\}$. Prove that there exists a holomorphic function $F : \mathbb{C} - \{0\} \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \mathbb{C} - \{0\}$. State clearly any results you use from lectures.

SECTION B

7. (a) Explain why if $f : D \to \mathbb{C}$ is a holomorphic function on a domain D with $f(x+iy) = u(x,y) + iv(x,y)$, then u is a harmonic function.

   (b) Let $u(x,y) = e^x x \cos y - e^x y \sin y$. Find a harmonic function $v : \mathbb{C} \to \mathbb{R}$ such that $f = u + iv$ is a holomorphic function on $\mathbb{C}$.

   (c) Let $f = u + iv$ be the function from part b). Calculate the complex derivative $f'(0)$.

8. (a) Find a Möbius transformation taking the region $\mathcal{R}_1 = \{z : |z| < 1, \mathrm{Im(z)} < 0\}$ (the lower half of the unit disc) to the upper half plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$.

   (b) Find a conformal map that maps the region $\mathcal{R}_1$ to $\mathcal{R}_2 = \{z : |z| < 1\} - \mathbb{R}_{\leq 0}$ (the unit disc with the non positive reals removed).

   (c) Find the image of $\mathcal{R}_2$ under the principal branch of log.

9. For $0 < \epsilon < R$, consider the closed contour

$$\gamma_{\epsilon,R} = L_2 + \widetilde{C_\epsilon} + L_1 + C_R,$$

where L_2 is the straight line running from $-R$ to $-\epsilon$, $\widetilde{C_\epsilon}(\theta) = \epsilon e^{-i\theta}$, $\theta \in [-\pi, 0]$, L_1 is the straight line running from ϵ to R, and $C_R(\theta) = Re^{i\theta}$, $\theta \in [0, \pi]$.

   (a) Let $g(z) = \frac{e^{iz}}{z}$. Calculate

$$\int_{\gamma_{\epsilon,R}} g(z) dz.$$

   (b) Show that $\lim_{R\to\infty} \int_{C_R} g(z)dz = 0$. You may use results from lectures provided they are stated clearly.

   (c) Show that $\lim_{\epsilon\to 0} \int_{\tilde{C}_\epsilon} g(z)\, dz = -\pi i$. Again, you may use results from lectures provided they are stated clearly.

   (d) By integrating $g(z)$ over $\gamma_{\epsilon,R}$ and using a), b) and c) show that

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

10. (a) State Cauchy's residue theorem for simple closed contours.

   (b) By using the substitution $z = e^{i\theta}$, or otherwise, evaluate the integral

$$\int_0^{2\pi} \frac{1}{1 + 3\cos^2(\theta)} d\theta.$$