EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2031-WE01

Title:
Analysis in Many Variables II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Sketch the three-dimensional vector field

$$\mathbf{A}(x,y,z) = (x-1)\mathbf{e}_1 + (y-1)\mathbf{e}_2 + (z-1)\mathbf{e}_3$$

in the $z = 1$ plane.

(b) Compute the divergence of $\mathbf{A}$ and comment on it in relation to your sketch.

(c) Compute the gradient, ∇f, of $f(x,y,z) = x^2 + xy + y^2 \sin z$.

(d) In which direction is this function f increasing the fastest at the point $(1, 1, \pi)$, and what is the vector equation for the normal line to the surface defined by $f(x,y,z) = 2$ at this point?

2. (a) Consider the transformation from polar coordinates (r,θ) back to cartesian coordinates (x,y):

$$\mathbf{x}(r,\theta) \;=\; \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix} \;=\; \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

By calculating the Jacobian $J(\mathbf{x})$, show whether the transformation is orientation preserving or not.

(b) Let $F(r,\theta) = f(x(r,\theta), y(r,\theta))$ where $f(x,y) = e^{(x^2-y^2)}$. Use the chain rule to calculate $\frac{\partial F}{\partial \theta}$, expressing your answer in terms of r and θ only.

3. Let $\mathbf{x}$ be the position vector in three dimensions, with $r = ||\mathbf{x}||$, and let $\mathbf{a}$ be a constant vector.

(a) Using index notation, show that grad $r = \mathbf{x}/r$.

(b) Using index notation, calculate the curl of $(\mathbf{a} \times \mathbf{x}/r^2)$.

4. (a) Compute the double integral

$$\int_0^{\pi} \int_0^{x} \frac{\sin(x)}{x}\, dy\, dx\,.$$

(b) The sine integral function $\mathrm{Si}(x)$ is defined by

$$\mathrm{Si}(x) = \int_0^{x} \frac{\sin(t)}{t}\, dt\,.$$

By reversing the order of integration in part (a) and using a theorem the name of which you should state, prove that

$$\int_0^{\pi} \mathrm{Si}(y)\, dy = A\,\pi\,\mathrm{Si}(\pi) - B$$

where A and B are two integers which you should determine.

5. (a) Let $\mathbf{u}(\mathbf{x})$ be the vector field in $\mathbb{R}^3$ defined by

$$\mathbf{u}(x, y, z) = 2xy\sin(z)\,\mathbf{e}_1 + (x^2 + 2y)\sin(z)\,\mathbf{e}_2 + (x^2y + y^2)\cos(z)\,\mathbf{e}_3\,.$$

Find a scalar field $f(x, y, z)$ such that $\mathbf{u} = \nabla f$.

(b) Using your answer to part (a) or otherwise, compute the line integral

$$\int_C \mathbf{u} \cdot d\mathbf{x}$$

where C is the straight line running from the origin to the point $(2, 3, 3\pi/2)$.

(c) Now let $\mathbf{v}(\mathbf{x})$ be the vector field

$$\mathbf{v}(x, y, z) = 2xy\sin(z)\,\mathbf{e}_1 + 2xy\sin(z)\,\mathbf{e}_2 + x^2y\cos(z)\,\mathbf{e}_3\,.$$

Is it possible to write $\mathbf{v}$ as $\mathbf{v} = \nabla k$ for some scalar field $k(x, y, z)$? Justify your answer.

6. By integrating both sides against an arbitrary test function, find the coefficients a, b, c, d, g and h in the following generalised function identities:

(a) $(x^2 - 1)\delta(x + 2) = a\delta(x - b)\,;$

(b) $(x^2 - 1)\delta'(x + 2) = c\delta(x - d) + g\delta'(x - d)\,;$

(c) $\delta(x/2) = h\delta(x)\,.$

SECTION B

7. (a) Suppose $f(x, y)$ is a scalar function on $\mathbb{R}^2$. State sufficient conditions for a critical point $\mathbf{a}$ of f to be either (i) a local maximum, (ii) a local minimum, or (iii) a saddle point of f, in terms of the 2×2 Hessian matrix, which you should also define.

   (b) Show that the function

$$f(x, y) = 8x^4 - 6x^2y + y^2$$

   has a unique critical point $\mathbf{a}_0$ and give $\mathbf{a}_0$.

   (c) Show that in this case the Hessian is unable to determine whether the critical point $\mathbf{a}_0$ is a maximum, minimum or saddle point.

   (d) Show further that $\mathbf{a}_0$ is a local minimum of the restriction of f to *any* straight line through $\mathbf{a}_0$.

   (e) By considering suitably chosen quadratic curves through $\mathbf{a}_0$, show that, in spite of the results of part (d), the point $\mathbf{a}_0$ is *not* a local minimum of f. Comment on your answer.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function on $\mathbb{R}^n$.

   (a) Give the definition of the partial derivative $\frac{\partial f}{\partial x_i}$ at $\mathbf{x} = \mathbf{a}$ as a limit.

   (b) Suppose f is differentiable at $\mathbf{a}$, so that $\mathbf{v}(\mathbf{a})$ exists such that

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = \mathbf{h} \cdot \mathbf{v}(\mathbf{a}) + R(\mathbf{h})$$

   State how $R(\mathbf{h})$ must behave as $\mathbf{h} \to \mathbf{0}$ in order to complete the above definition of differentiability.

   (c) Using the definitions from part (a) and part (b), prove that $\mathbf{v}(\mathbf{a}) = \nabla f(\mathbf{a})$.

   (d) What is meant by the statement that f is continuously differentiable at a point $\mathbf{a} \in \mathbb{R}^n$? State (but don't prove) a theorem relating continuous differentiability to differentiability.

   (e) The function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x, y) = (|x - 2| + 2)y$. Determine whether or not g is differentiable at the following points:

$$\text{(i)}\ \ (x, y) = (0, 0)\ , \qquad \text{(ii)}\ \ (x, y) = (2, 2)\ , \qquad \text{(iii)}\ \ (x, y) = (2, 0)\ .$$

   In each case you should fully justify your claim.

9. (a) State Stokes' theorem.

(b) A surface S is specified by the equation

$$z = \sin(x)\,\cos(y)\ , \qquad 0 \le x \le \pi\,, \quad 0 \le y \le \pi/2\,.$$

Draw a rough sketch of S, and find a formula for the unit upward-pointing (that is, with positive z component) normal $\widehat{\mathbf{n}}$ to S at a general point on S. (For the sketch, it may help to start by thinking what values z takes around the boundary of S.)

(c) Without using Stokes theorem, compute the surface integral

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$$

where S is the surface defined in part (a) of this question, and $\mathbf{F}$ is the vector field $\mathbf{F}(x,y,z) = z\,\mathbf{e}_1 + y\,\mathbf{e}_3$.

(d) Again without using Stokes' theorem, evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{x}$ where C is the boundary of S, taken anticlockwise when viewed from above, and $\mathbf{F}$ is as in part (c).

(e) What is the value of $\oint_C \mathbf{G} \cdot d\mathbf{x}$, if C is the same curve as in part (d), and $\mathbf{G}(x,y,z) = (y+z)\,\mathbf{e}_1 + (x+z)\,\mathbf{e}_2 + 2y\,\mathbf{e}_3$? (Hint: consider $\nabla \times (\mathbf{G} - \mathbf{F})$ first.)

10. For $t > 0$ the temperature $T(x,t)$ in a metal bar of length L satisifes the heat equation

$$\kappa^{-2}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \qquad (t > 0\,,\ 0 < x < L)$$

where κ is a constant. Note that κ is not necessarily equal to 1. The boundary conditions are such that for all $t > 0$,

$$T(0,t) = T(L,t) = 0\,.$$

Use the method of separation of variables to find a solution of this equation for $t > 0$ which satisfies

$$T(x,0) = T_0(x) = \frac{3x}{L}\left(1 - \frac{x}{L}\right) \qquad (0 < x < L)$$

at $t = 0$. Using your solution, show that as $t \to \infty$ the temperature at the mid-point of the bar decays to zero as

$$T(L/2,t) \sim \alpha \exp(-\beta t)$$

where α and β are two constants which you should determine.

If, rather than being an even function under $x \to L - x$, the initial condition was changed to be odd, so that $T_0(L-x) = -T_0(x)$, what would you expect the value of β to be? You do not need to give a detailed calculation, but you should justify your answer.