Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2041-WE01

Title:

Stastical Concepts II

Time Allowed:	3 hours	
Additional Material provided:	Tables: Normal distribution, t-distribution, chi-squared distribution, signed-rank test statistic, rank-sum test statistic.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. An independent and identically distributed sample of size n, $\underline{x} = (x_1, \ldots, x_n)$, is drawn from a Poisson distribution with parameter λ.

   (The Poisson distribution, with parameter λ, has probability mass function,

   $$\mathrm{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, \ldots)$$

   For this sampling problem, answer the following questions.

   (a) Find a sufficient statistic for λ and justify your answer.

   (b) Find the maximum likelihood estimator $\hat{\lambda}$ for λ. Show that it depends on the sample only through the value of the sufficient statistic.

   (c) Explain why the maximum likelihood estimator depends on the sample only through the value of the sufficient statistic for any sampling problem for which a sufficient statistic exists.

2. Suppose that a random quantity, X, is considered to have an exponential distribution, with parameter λ.

   (The probability density function of the exponential distribution is

   $$f(x|\lambda) = \lambda \mathrm{e}^{-\lambda x}, \ x > 0.)$$

   (a) Find the expectation and variance of X.

   (b) In order to test the value of λ, an independent sample of size n, $X_1, \ldots, X_n$, is taken. We wish to test the null hypothesis, H_0, that $\lambda = 1$ against the alternative hypothesis, H_1, that $\lambda = 2$, at significance level α. Find the most powerful test of H_0 against H_1. (State, without proof, any results that you use to demonstrate that the test is most powerful.)

   (c) Suppose that $n = 50$, and the chosen significance level is 0.05. Find, approximately, the critical value for this test, and the power of the test.

3. A particular test of proficiency at a particular skill classifies each individual taking the test as having high or low skill. In a random sample of 200 tested individuals from a given target group, the gender of each individual was also recorded. The data was as follows.

	High	Low
Male	42	45
Female	64	49

   Test whether there is an association between skill and gender. State carefully the hypothesis that is being tested and explain the calculations that are carried out in order to apply the test.

4. Suppose that three random variables X_1, X_2, X_3 have a continuous joint distribution with probability density function

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = 8x_1x_2x_3 \qquad \text{for } 0 < x_i < 1, i = 1, 2, 3$$

and 0 otherwise. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, and $Y_3 = X_1X_2X_3$.

(a) Show that the joint probability density function of Y_1, Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = 4\frac{y_2}{y_1} \qquad \text{for } 0 < y_2 < y_1 < 1$$

and 0 otherwise.

(b) Derive the conditional probability density function of $Y_2|Y_1$.

(c) Are Y_1 and Y_2 independent?

5. Soil respiration is a measure of microbial activity in soil, which affects plant growth. In one study, soil cores were taken from two types of locations in a forest: under openings in the forest canopy ("Open"), and under heavy tree growth ("Growth"). The level of soil respiration was assessed by measuring the amount of carbon dioxide given off by each soil core (in mol CO_2/g soil/hr), producing the data below.

Growth	17	20	170	315	22	190	64		
Open	24	29	21	13	16	23	18	15	6

(a) Define the rank sum test statistic for testing the null hypothesis of no difference between two population distributions, based on independent random samples of sizes n and m, from each population.

(b) Apply the rank sum test to perform an exact test of whether soil respiration differs between the two locations.

(c) Repeat the test performed in (b) using an appropriate large sample approximation to the null distribution of the nonparametric test statistic. Comment on the validity of this approximation for these data.

6. Suppose that the lengths in millimetres of metal fibres produced by a certain process have a normal distribution for which the mean and variance are both unknown. Suppose that the lengths of 7 fibres selected at random are measured and are

$$1267, 1262, 1267, 1263, 1258, 1263, 1268$$

(a) For this sample, find the approximate probability that the sample variance is at least twice the value of the population variance.

(b) Derive a general expression for a $100(1 - \alpha)\%$ confidence interval for the population variance.

(c) Find a 95% confidence interval for the population variance.

(d) Calculate a 95% confidence interval for the population mean.

SECTION B

7. A certain quantity x can be measured for each member of a population of size N. The values of x in the population are $x_1, \dots, x_N$. A random sample $Y_1, \dots, Y_n$, of size n, is selected without replacement from the population.

   (a) Define the population mean, μ, and the population variance σ^2.

   (b) Show that the expected value of the sample variance, s^2 is given by
   $$E(s^2) = \frac{N}{N-1}\sigma^2$$
   Hence derive an unbiased estimator, $s^2_{\overline{Y}}$, for $\sigma^2_{\overline{Y}}$, the variance of the sample mean, $\overline{Y}$.
   (You may quote, without proof, the value of $\sigma^2_{\overline{Y}}$.)

   (c) In the special case of binary values, where each x_i is zero or one, denote by p the proportion of the population for which $x = 1$ and by $\hat{p}$ the proportion of the sample for which $x = 1$. State the corresponding versions, $\sigma^2_{\hat{p}}$ of $\sigma^2_{\overline{Y}}$ and $s^2_{\hat{p}}$ of $s^2_{\overline{Y}}$ as functions of p and of $\hat{p}$ respectively.

   (d) In a particular population, of size $N = 8$, the value of x for five individuals is one, and for the remaining three individuals the value is zero. A sample of size 2 is taken, without replacement. Evaluate the sampling distribution of $\hat{p}$ and of $s^2_{\hat{p}}$. Hence evaluate directly the variance of $\hat{p}$ and the mean of $s^2_{\hat{p}}$. Confirm that these answers agree with the general results stated in part (c).

8. A particular random quantity, X, can take three possible values, namely 1, 2, 3.

   The probability that X takes value i is $p_i, i = 1, 2, 3$.

   An independent random sample, of size n, is taken from the distribution of X. The number of members of the sample for which $X = i$ is $x_i, i = 1, 2, 3$.

   (a) Suppose that $p_1 = p_2 = p, p_3 = 1-2p$. Find the maximum likelihood estimator $\hat{p}$ for p.

   (b) Find the expectation and variance of $\hat{p}$.

   (c) Find Fisher's information for p.
   State the general relationship between Fisher's information and the large sample properties of the maximum likelihood estimator. Explain how to use Fisher's information to construct an approximate large sample 95% confidence interval for p.

   (d) Comparing the approximate assessment of the properties of $\hat{p}$ based on the calculations in part (c) with the precise calculation in part (b), explain carefully what the above analysis tells you about the optimality of the maximum likelihood estimator for this problem.

   (e) Suppose that we wish to test the null hypothesis that $p_1 = p_2 = p$ against the alternative hypothesis that $p_1 \neq p_2$. Find the generalised likelihood ratio test statistic for this problem. Explain how to determine critical values for the statistic for large n.

9. (a) A common observation in ecology is that species diversity decreases as the latitude of the location (i.e. the distance from the equator) increases. To investigate this further, a series of counts of unique bird species were conducted, each within in a 15-mile diameter area centred on 17 different locations at different latitudes. The counts were all conducted on the same day in 2005 in eastern coastal states of the USA. The data obtained were as follows:

Latitude, x	39.22	38.80	39.47	38.96	38.60	38.58	39.73	38.03	38.90
Count, y	128	137	108	118	135	94	113	118	96
Latitude, x	39.53	39.13	38.32	38.33	38.37	37.20	37.97	37.67	
Count, y	98	121	152	108	118	157	125	114	

The following summary statistics were calculated for these values: $\bar{x} = 38.63588$, $\bar{y} = 120$, $\sum_{i=1}^{17} x_i^2 = 25383.99$, $\sum_{i=1}^{17} y_i^2 = 249918$, and $\sum_{i=1}^{17} x_i y_i = 78726.39$.

i. Calculate the least-squares regression line. Find and interpret the R^2 value, and assess the quality of the regression.

ii. Use the regression line to obtain a 95% prediction interval for the species count in Durham, which has a latitude 54.775. Comment on the validity and reliability of your prediction.

(b) Weighted linear regression is a variation of the simple linear regression model

$$y_i = \gamma_0 + \gamma_1 x_i + \varepsilon_i$$

where the errors ε_i have mean 0 and are independent, but $\mathrm{Var}(\varepsilon_i) = \sigma^2/w_i$, where the $w_i > 0$ are known constant weights for each $i = 1, \ldots, n$.

i. Explain why it would be inappropriate to apply the standard least-squares approach to simple linear regression in this problem.

ii. The regression relationship for each y_i is transformed by multiplication by $\sqrt{w_i}$ to give:

$$\sqrt{w_i} y_i = \gamma_0 \sqrt{w_i} + \gamma_1 \sqrt{w_i} x_i + \sqrt{w_i} \varepsilon_i$$

Verify that this new formulation of model satisfies the standard assumptions of simple linear regression.

iii. Apply the method of least squares to the model in part (b)(ii) to find expressions for the least-squares estimates $\hat{\gamma}_0$ and $\hat{\gamma}_1$ of γ_0 and γ_1.

iv. Consider the special case where $w_i = 1$ for all $i = 1, \ldots, n$. Give expressions for $\hat{\gamma}_0$ and $\hat{\gamma}_1$ in this case, and comment on your results.

10. Suppose that we have an independent sample of n observations, $x_1, \ldots, x_n$, from a normal distribution, with unknown mean w and known precision $r > 0$ (so that the variance is $\sigma^2 = 1/r$). Let the prior distribution for w be a normal distribution with mean μ and precision $s > 0$. The probability density function of the normal distribution with mean w and precision $r > 0$ is, for real-valued x,

$$f(x) = \sqrt{\frac{r}{2\pi}} e^{-\frac{r}{2}(x-w)^2}$$

(a) Show that this family of prior distributions is conjugate for sampling from a normal distribution with known precision, and that the corresponding posterior distribution for w is normal with mean μ_1 and precision s_1, where, if $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,

$$\mu_1 = \frac{s\mu + nr\bar{x}}{s + nr}, \qquad s_1 = s + nr$$

You may use, without proof, the following result

$$\sum_{i=1}^{n}(x_i - a)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - a)^2.$$

(b) In a study of the calorie content of various foods, Allison et al (1993) sampled 20 food items and calculated the difference between the advertised calorie contents on the packaging label to the calorie content determined in a laboratory test.

Suppose that the differences in calorie counts (laboratory calories − advertised calories) can be considered to be a random sample from a normal distribution with an unknown mean value w and variance 10. Suppose that prior beliefs about w are represented by a normal distribution with mean 0 and variance 2. If the average difference in calorie counts for the collection of 20 food items is -0.95, derive the corresponding central 95% posterior credible interval for w.

(c) i. Consider what happens to the posterior distribution in the problem of part (a) for large sample sizes, by evaluating what happens to the posterior parameters when n becomes very large.

ii. State the form of the limiting posterior distribution for the parameter of a general likelihood as the sample size increases. Find the limiting posterior form for sampling from the normal distribution with known precision, using this result. (You may assume, without proof, that the sample mean is the maximum likelihood estimate for w for normal samples.)

iii. Compare the limiting forms derived in (c)(i) and (c)(ii).

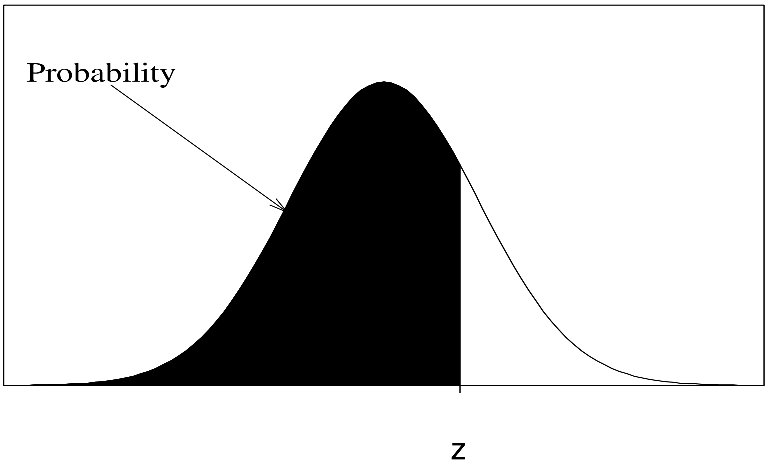
Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z, i.e. $\Phi(z)$.

For $z > 3$,

$$1 - \Phi(z) \approx \frac{1}{\sqrt{2\pi}z} e^{-\frac{1}{2}z^2}$$

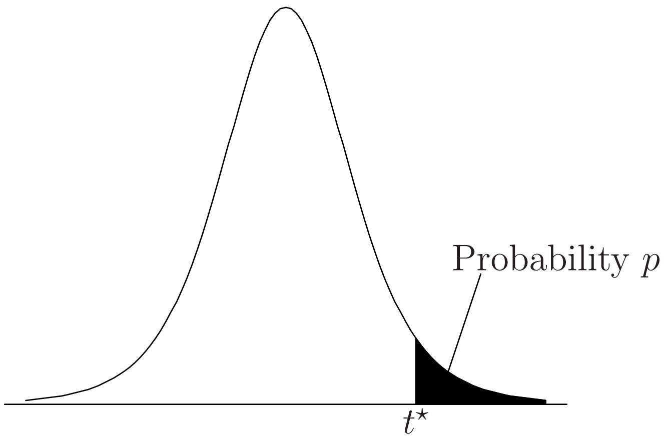
is accurate to within 10% of the true value.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Probabilities for the t-distribution

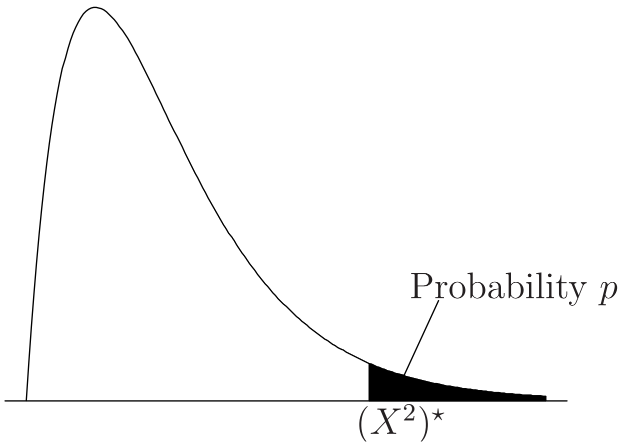
Table entry for p and C is the point $t^\star$ with probability p lying above it and probability C lying between $-t^\star$ and $t^\star$



	Tail probability p											
df	.25	.2	.15	.1	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.895	31.821	63.657	127.321	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70 %	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

Probabilities for the χ^2-distribution

Table entry for p is the point $(X^2)^\star$ with probability p lying above it



	Tail probability p											
df	.995	.975	.25	.2	.1	.05	.025	.01	.005	.0025	.001	.0005
1	0.000039	0.00098	1.32	1.64	2.71	3.84	5.02	6.63	7.88	9.14	10.83	12.12
2	0.010	0.051	2.77	3.22	4.61	5.99	7.38	9.21	10.60	11.98	13.82	15.20
3	0.072	0.22	4.11	4.64	6.25	7.81	9.35	11.34	12.84	14.32	16.27	17.73
4	0.21	0.48	5.39	5.99	7.78	9.49	11.14	13.28	14.86	16.42	18.47	20.00
5	0.41	0.83	6.63	7.29	9.24	11.07	12.83	15.09	16.75	18.39	20.52	22.11
6	0.68	1.24	7.84	8.56	10.64	12.59	14.45	16.81	18.55	20.25	22.46	24.10
7	0.99	1.69	9.04	9.80	12.02	14.07	16.01	18.48	20.28	22.04	24.32	26.02
8	1.34	2.18	10.22	11.03	13.36	15.51	17.53	20.09	21.95	23.77	26.12	27.87
9	1.73	2.70	11.39	12.24	14.68	16.92	19.02	21.67	23.59	25.46	27.88	29.67
10	2.16	3.25	12.55	13.44	15.99	18.31	20.48	23.21	25.19	27.11	29.59	31.42
11	2.60	3.82	13.70	14.63	17.28	19.68	21.92	24.72	26.76	28.73	31.26	33.14
12	3.07	4.40	14.85	15.81	18.55	21.03	23.34	26.22	28.30	30.32	32.91	34.82
13	3.57	5.01	15.98	16.98	19.81	22.36	24.74	27.69	29.82	31.88	34.53	36.48
14	4.07	5.63	17.12	18.15	21.06	23.68	26.12	29.14	31.32	33.43	36.12	38.11
15	4.60	6.26	18.25	19.31	22.31	25.00	27.49	30.58	32.80	34.95	37.70	39.72
16	5.14	6.91	19.37	20.47	23.54	26.30	28.85	32.00	34.27	36.46	39.25	41.31
17	5.70	7.56	20.49	21.61	24.77	27.59	30.19	33.41	35.72	37.95	40.79	42.88
18	6.26	8.23	21.60	22.76	25.99	28.87	31.53	34.81	37.16	39.42	42.31	44.43
19	6.84	8.91	22.72	23.90	27.20	30.14	32.85	36.19	38.58	40.88	43.82	45.97
20	7.43	9.59	23.83	25.04	28.41	31.41	34.17	37.57	40.00	42.34	45.31	47.50
21	8.03	10.28	24.93	26.17	29.62	32.67	35.48	38.93	41.40	43.78	46.80	49.01
22	8.64	10.98	26.04	27.30	30.81	33.92	36.78	40.29	42.80	45.20	48.27	50.51
23	9.26	11.69	27.14	28.43	32.01	35.17	38.08	41.64	44.18	46.62	49.73	52.00
24	9.89	12.40	28.24	29.55	33.20	36.42	39.36	42.98	45.56	48.03	51.18	53.48
25	10.52	13.12	29.34	30.68	34.38	37.65	40.65	44.31	46.93	49.44	52.62	54.95
26	11.16	13.84	30.43	31.79	35.56	38.89	41.92	45.64	48.29	50.83	54.05	56.41
27	11.81	14.57	31.53	32.91	36.74	40.11	43.19	46.96	49.64	52.22	55.48	57.86
28	12.46	15.31	32.62	34.03	37.92	41.34	44.46	48.28	50.99	53.59	56.89	59.30
29	13.12	16.05	33.71	35.14	39.09	42.56	45.72	49.59	52.34	54.97	58.30	60.73
30	13.79	16.79	34.80	36.25	40.26	43.77	46.98	50.89	53.67	56.33	59.70	62.16
40	20.71	24.43	45.62	47.27	51.81	55.76	59.34	63.69	66.77	69.70	73.40	76.09
50	27.99	32.36	56.33	58.16	63.17	67.50	71.42	76.15	79.49	82.66	86.66	89.56
60	35.53	40.48	66.98	68.97	74.40	79.08	83.30	88.38	91.95	95.34	99.61	102.69
80	51.17	57.15	88.13	90.41	96.58	101.88	106.63	112.33	116.32	120.10	124.84	128.26
100	67.33	74.22	109.14	111.67	118.50	124.34	129.56	135.81	140.17	144.29	149.45	153.17

Values for the Mann-Whitney-Wilcoxon Test

Reject the hypothesis of identical populations if the test statistic is *less than* the value T_L shown in the following table or *greater than* the value T_U where

$$T_U = n_1(n_1 + n_2 + 1) - T_L$$

		$\mathbf{n_2}$								
$\alpha \leq$ **.01**		2	3	4	5	6	7	8	9	10
	2	–	–	–	–	–	–	–	–	–
	3	–	–	–	–	–	–	–	7	7
	4	–	–	–	–	11	11	12	12	13
	5	–	–	–	16	17	17	18	19	20
$\mathbf{n_1}$	6	–	–	22	23	24	25	26	27	28
	7	–	–	29	30	32	33	35	36	38
	8	–	–	38	39	41	43	44	46	48
	9	–	46	47	49	51	53	55	57	59
	10	–	56	58	60	62	65	67	69	72

		$\mathbf{n_2}$								
$\alpha \leq$ **.05**		2	3	4	5	6	7	8	9	10
	2	–	–	–	–	–	–	4	4	4
	3	–	–	–	7	8	8	9	9	10
	4	–	–	11	12	13	14	15	15	16
	5	–	16	17	18	19	21	22	23	24
$\mathbf{n_1}$	6	–	23	24	25	27	28	30	32	33
	7	–	30	32	34	35	37	39	41	43
	8	37	39	41	43	45	47	50	52	54
	9	46	48	50	53	56	58	61	63	66
	10	56	59	61	64	67	70	73	76	79

		$\mathbf{n_2}$								
$\alpha \leq$ **.10**		2	3	4	5	6	7	8	9	10
	2	–	–	–	4	4	4	5	5	5
	3	–	7	7	8	9	9	10	11	11
	4	–	11	12	13	14	15	16	17	18
	5	16	17	18	20	21	22	24	25	27
$\mathbf{n_1}$	6	22	24	25	27	29	30	32	34	36
	7	29	31	33	35	37	40	42	44	46
	8	38	40	42	45	47	50	52	55	57
	9	47	50	52	55	58	61	64	67	70
	10	57	60	63	67	70	73	76	80	83

Values for the Wilcoxon signed-rank Test

Reject the hypothesis of identical populations if the test statistic is *less than* the value T shown in the following table.

Sample size	Level of significance for a two-tailed test			
n	10%	5%	2%	1%
5	1	–	–	–
6	3	1	–	–
7	4	3	1	–
8	6	4	2	1
9	9	6	4	2
10	11	9	6	4
11	14	11	8	6
12	18	14	10	8
13	22	18	13	10
14	26	22	16	13
15	31	26	20	16
16	36	30	24	20
17	42	35	28	24
18	48	41	33	28
19	54	47	38	33
20	61	53	44	38
21	68	59	50	43
22	76	66	56	49
23	84	74	63	55
24	92	82	70	62
25	101	90	77	69