

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH2051-WE01

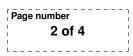
Title:

Numerical Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates: Credit will be given for: the best FOUR answers from Section A and the best THREE answers from Section B. Questions in Section B carry TWICE as many marks as those in Section A.

Revision:



SECTION A

- 1. (a) Define briefly what is meant by a floating-point number having (i) a finite precision and (ii) a finite range.
 - (b) A student tried to use the fact that $f(x) := (1 + 1/x)^x \to e$ as $x \to \infty$ to compute $e \doteq 2.718281828459045235 \cdots$ in Python. With $x_n = 2^n$, he found

$$f(x_0) = 2$$

$$f(x_1) = 2.25$$

$$\vdots$$

$$f(x_{52}) = 2.718281828459045$$

$$f(x_{53}) = 1.0$$

with the rest of the iterates being 1.0. Explain this result.

- (c) Define what is meant by the *machine epsilon* of a floating-point representation and estimate (within a factor of 2) the machine epsilon in the computation above.
- 2. (a) Show that the equation

$$\exp(-x) - a^2 x^2 = 0 \tag{1}$$

has exactly one positive solution for every a > 0.

- (b) Given $a > \frac{1}{2}$, one seeks to find the positive solution of (1) using the iteration $x_{n+1} = \exp(-x_n/2)/a$. Show that this iteration converges for all x_0 in some interval in \mathbb{R}_+ and find one such interval.
- 3. (a) Show that the backward differentiation formula (BDF)

$$f'(x) \simeq \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}$$

approximates f'(x) with order 2.

(b) Given the following data about some f(x),

$$f(0) = 1.0$$

$$f(0.0125) = 0.9929375954$$

$$f(0.025) = 0.9861739631$$

$$f(0.05) = 0.9735042655$$

$$f(0.1) = 0.9513507698$$

compute the BDF approximations to f'(0) with |h| = 0.05, 0.025 and 0.0125.

- (c) What is the best (highest order) approximation to f'(0) that one can compute with the data above? Justify your answer briefly and give the numerical value.
- 4. Find nodes $\{t_0, t_1, t_2\} \subset [0, 1]$ and coefficients $\{\sigma_0, \sigma_1, \sigma_2\}$ such that the quadrature

$$\sigma_0 f(t_0) + \sigma_1 f(t_1) + \sigma_2 f(t_2) = \int_0^1 f(t) \, \mathrm{d}t$$

is exact for every $f \in \mathcal{P}_5$. Hint: symmetry.

- 5. (a) State the properties that define a *norm*.
 - (b) Give the definition of the matrix *row-sum* norm and show that it is indeed a norm.
 - (c) Let

$$A = \left(\begin{array}{rrrr} 3 & 1 & -4 \\ 1 & -5 & 9 \\ -2 & -6 & 5 \end{array}\right).$$

Compute $||A||_{\infty}$ and find a vector v such that $||Av||_{\infty} = ||A||_{\infty} ||v||_{\infty}$.

- 6. Let $\phi_j \in \mathcal{P}_j$ be a set of orthogonal polynomials in [0, 1] with weight w(x) = 1.
 - (a) Compute the first four members $\{\phi_0, \dots, \phi_3\}$ of these polynomials.
 - (b) Prove for all $k \in \{0, 1, \dots\}$ that $\phi_k(x)$ is symmetric about $x = \frac{1}{2}$ if k is even, and antisymmetric if k is odd.
 - (c) Prove that $\phi_k(x) = 0$ has exactly k distinct solutions in [0, 1].

SECTION B

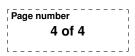
- 7. (a) Define what is meant by the order of convergence of a sequence (x_k) .
 - (b) Give an explicit sequence that converges (say, to 0) with order 3.
 - (c) Consider a convergent iteration $x_{k+1} = g(x_k)$ with $g'(x_*) = 0$ at the fixed point x_* . Show that if $g \in C^2$ one has

$$x_{k+1} - x_* = \frac{(x_k - x_*)^2}{2!} g''(\xi_k) \quad \text{for some } \xi_k \in \text{conv}\{x_k, x_*\}$$

and that the iteration converges with order at least 2.

- (d) Write down the Newton-Raphson iteration to solve f(x) = 0. Give a sufficient condition that the order of this iteration is *exactly* 2.
- 8. (a) Show that a divided difference is symmetric in its nodes, i.e. if $(y_j)_{j=0}^n$ is a permutation of $(x_j)_{j=0}^n$, then $[y_0, \dots, y_n]f = [x_0, \dots, x_n]f$.
 - (b) With nodes $(x_0, x_1, x_2) = (-1, 0, 1)$, one computes $[x_0]f = -4$, $[x_0, x_1]f = 3$ and $[x_0, x_1, x_2]f = 2$. Now one is given a new node $x_3 = -2$ with $[x_3]f = -3$. Write down the interpolating cubic polynomial $p_3(x)$ and compute $p_3(0.5)$.
 - (c) State (do not prove) Cauchy's error formula for polynomial interpolation and prove that with the nodes (-2, -1, 0, 1) above, one has the error estimate

$$|f(x) - p_3(x)| \le \frac{1}{24} \sup_{x \in [-2,1]} |f^{(4)}(x)|.$$





- 9. (a) Prove that if a non-singular matrix A can be written as A = MU (where M is lower and U is upper triangular) and the diagonal elements of M are given, then M and U are unique.
 - (b) If one is given $n \times n$ matrices U and L, upper and lower triangular with $l_{jj} = 1$, such that LU = A, how many elementary arithmetic operations are needed to solve Ax = b? Derive the leading order term (highest power in n) only.
 - (c) Perform LU decomposition on

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 4 & 11 & 11 & 10 \\ 6 & 29 & 45 & 25 \\ 6 & 19 & 27 & 21 \end{pmatrix}.$$

- (d) Solve $Ax = b = (15, 29, 34, 36)^{\mathrm{T}}$ given that $x_4 = 1$.
- (e) Which singular matrices can be written as A = LU, where U is upper triangular and L lower triangular with $L_{ii} = 1$? All, some or none? Justify your answer.
- 10. (a) Define what is meant by an interpolatory quadrature formula $\mathcal{I}_n(f)$ and write down (do not prove) an error bound for it.
 - (b) Given the closed Newton–Cotes formula

$$\mathcal{I}_1(f) = \frac{b-a}{2}[f(a) + f(b)],$$

write down the formula for the composite quadrature $C_{1,m}(f)$ and obtain an $O(m^{-2})$ error bound for it.

(c) Given the closed Newton–Cotes formula

$$\mathcal{I}_2(f) = \frac{b-a}{6} \Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b) \Big],$$

write down the formula for the composite quadrature $C_{2,m}(f)$ and obtain an $O(m^{-3})$ error bound for it.

(d) Now take $f(x) = (\sin x)/x$ with f(0) := 0, and a = 0 and b = 10. One computes $C_{1,50} \doteq 1.6580858979$, $C_{1,100} \doteq 1.6582821964$ and $C_{1,200} \doteq 1.6583312464$. Compute $C_{2,50}$ and $C_{2,100}$. Given that the error in these quadratures all have even orders, compute the most accurate approximation obtainable from the data.

Possibly useful integrals:

$$\int_0^1 |x(x-1)| \, \mathrm{d}x = \frac{1}{6} \quad \text{and} \quad \int_0^2 |x(x-1)(x-2)| \, \mathrm{d}x = \frac{1}{2}.$$