EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2051-WE01

Title:
Numerical Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Define briefly what is meant by a floating-point number having (i) a finite precision and (ii) a finite range.

   (b) A student tried to use the fact that $f(x) := (1 + 1/x)^x \to \mathrm{e}$ as $x \to \infty$ to compute $\mathrm{e} \doteq 2.718281828459045235 \cdots$ in Python. With $x_n = 2^n$, he found

$$
\begin{aligned}
f(x_0) &= 2 \\
f(x_1) &= 2.25 \\
&\vdots \\
f(x_{52}) &= 2.718281828459045 \\
f(x_{53}) &= 1.0
\end{aligned}
$$

   with the rest of the iterates being 1.0. Explain this result.

   (c) Define what is meant by the *machine epsilon* of a floating-point representation and estimate (within a factor of 2) the machine epsilon in the computation above.

2. (a) Show that the equation

$$\exp(-x) - a^2x^2 = 0 \tag{1}$$

   has exactly one positive solution for every $a > 0$.

   (b) Given $a > \frac{1}{2}$, one seeks to find the positive solution of (1) using the iteration $x_{n+1} = \exp(-x_n/2)/a$. Show that this iteration converges for all x_0 in some interval in $\mathbb{R}_+$ and find one such interval.

3. (a) Show that the backward differentiation formula (BDF)

$$f'(x) \simeq \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}$$

   approximates $f'(x)$ with order 2.

   (b) Given the following data about some $f(x)$,

$$
\begin{aligned}
f(0) &= 1.0 \\
f(0.0125) &= 0.99293\,75954 \\
f(0.025) &= 0.98617\,39631 \\
f(0.05) &= 0.97350\,42655 \\
f(0.1) &= 0.95135\,07698
\end{aligned}
$$

   compute the BDF approximations to $f'(0)$ with $|h| = 0.05$, 0.025 and 0.0125.

   (c) What is the best (highest order) approximation to $f'(0)$ that one can compute with the data above? Justify your answer briefly and give the numerical value.

4. Find nodes $\{t_0, t_1, t_2\} \subset [0, 1]$ and coefficients $\{\sigma_0, \sigma_1, \sigma_2\}$ such that the quadrature

$$\sigma_0 f(t_0) + \sigma_1 f(t_1) + \sigma_2 f(t_2) = \int_0^1 f(t)\,\mathrm{d}t$$

   is exact for every $f \in \mathcal{P}_5$. Hint: symmetry.

5. (a) State the properties that define a *norm*.

   (b) Give the definition of the matrix *row-sum* norm and show that it is indeed a norm.

   (c) Let

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 1 & -5 & 9 \\ -2 & -6 & 5 \end{pmatrix}.$$

   Compute $\|A\|_\infty$ and find a vector v such that $\|Av\|_\infty = \|A\|_\infty \|v\|_\infty$.

6. Let $\phi_j \in \mathcal{P}_j$ be a set of orthogonal polynomials in $[0,1]$ with weight $w(x) = 1$.

   (a) Compute the first four members $\{\phi_0, \cdots, \phi_3\}$ of these polynomials.

   (b) Prove for all $k \in \{0, 1, \cdots\}$ that $\phi_k(x)$ is symmetric about $x = \frac{1}{2}$ if k is even, and antisymmetric if k is odd.

   (c) Prove that $\phi_k(x) = 0$ has exactly k distinct solutions in $[0,1]$.

SECTION B

7. (a) Define what is meant by the *order of convergence* of a sequence (x_k).

   (b) Give an explicit sequence that converges (say, to 0) with order 3.

   (c) Consider a convergent iteration $x_{k+1} = g(x_k)$ with $g'(x_*) = 0$ at the fixed point x_*. Show that if $g \in C^2$ one has

$$x_{k+1} - x_* = \frac{(x_k - x_*)^2}{2!}\, g''(\xi_k) \quad \text{for some } \xi_k \in \operatorname{conv}\{x_k, x_*\}$$

   and that the iteration converges with order at least 2.

   (d) Write down the Newton–Raphson iteration to solve $f(x) = 0$. Give a sufficient condition that the order of this iteration is *exactly* 2.

8. (a) Show that a divided difference is symmetric in its nodes, i.e. if $(y_j)_{j=0}^n$ is a permutation of $(x_j)_{j=0}^n$, then $[y_0, \cdots, y_n]f = [x_0, \cdots, x_n]f$.

   (b) With nodes $(x_0, x_1, x_2) = (-1, 0, 1)$, one computes $[x_0]f = -4$, $[x_0, x_1]f = 3$ and $[x_0, x_1, x_2]f = 2$. Now one is given a new node $x_3 = -2$ with $[x_3]f = -3$. Write down the interpolating cubic polynomial $p_3(x)$ and compute $p_3(0.5)$.

   (c) State (do not prove) Cauchy's error formula for polynomial interpolation and prove that with the nodes $(-2, -1, 0, 1)$ above, one has the error estimate

$$|f(x) - p_3(x)| \le \frac{1}{24} \sup_{x \in [-2,1]} |f^{(4)}(x)|.$$

9. (a) Prove that if a non-singular matrix A can be written as $A = MU$ (where M is lower and U is upper triangular) and the diagonal elements of M are given, then M and U are unique.

   (b) If one is given $n \times n$ matrices U and L, upper and lower triangular with $l_{jj} = 1$, such that $LU = A$, how many elementary arithmetic operations are needed to solve $Ax = b$? Derive the leading order term (highest power in n) only.

   (c) Perform LU decomposition on

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 4 & 11 & 11 & 10 \\ 6 & 29 & 45 & 25 \\ 6 & 19 & 27 & 21 \end{pmatrix}.$$

   (d) Solve $Ax = b = (15, 29, 34, 36)^{\mathrm{T}}$ given that $x_4 = 1$.

   (e) Which singular matrices can be written as $A = LU$, where U is upper triangular and L lower triangular with $L_{jj} = 1$? All, some or none? Justify your answer.

10. (a) Define what is meant by an interpolatory quadrature formula $\mathcal{I}_n(f)$ and write down (do not prove) an error bound for it.

   (b) Given the closed Newton–Cotes formula

$$\mathcal{I}_1(f) = \frac{b-a}{2}[f(a) + f(b)],$$

   write down the formula for the composite quadrature $\mathcal{C}_{1,m}(f)$ and obtain an $O(m^{-2})$ error bound for it.

   (c) Given the closed Newton–Cotes formula

$$\mathcal{I}_2(f) = \frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big],$$

   write down the formula for the composite quadrature $\mathcal{C}_{2,m}(f)$ and obtain an $O(m^{-3})$ error bound for it.

   (d) Now take $f(x) = (\sin x)/x$ with $f(0) := 0$, and $a = 0$ and $b = 10$. One computes $\mathcal{C}_{1,50} \doteq 1.65808\,58979$, $\mathcal{C}_{1,100} \doteq 1.65828\,21964$ and $\mathcal{C}_{1,200} \doteq 1.65833\,12464$. Compute $\mathcal{C}_{2,50}$ and $\mathcal{C}_{2,100}$. Given that the error in these quadratures all have even orders, compute the most accurate approximation obtainable from the data.

Possibly useful integrals:

$$\int_0^1 |x(x-1)|\,\mathrm{d}x = \frac{1}{6} \quad \text{and} \quad \int_0^2 |x(x-1)(x-2)|\,\mathrm{d}x = \frac{1}{2}.$$