EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2071-WE01

Title:
Mathematical Physics II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. A particle of unit mass moves in a two-dimensional plane parametrised by the Cartesian coordinates (x, y) without friction.

   (a) Find the kinetic energy of the system in terms of the polar coordinates (r, θ), defined via $x = r\cos(\theta)$, $y = r\sin(\theta)$.

   (b) Assuming that there is a rotationally symmetric potential $V(r) = \frac{r^6}{6}$, write the Lagrangian for the system in polar coordinates, and derive the Euler-Lagrange equations of motion associated to these polar coordinates. (You do not need to solve them.)

   (c) Which coordinate is ignorable? Write the associated generalised momentum J, and show that it is conserved.

   (d) Find an equation of motion for r only (without θ or $\dot{\theta}$ appearing anywhere) in terms of the conserved charge J.

2. The energy for a system with Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ is defined to be

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L\,.$$

   (a) Show, using the Euler-Lagrange equations of motion, that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,.$$

   (b) Show that the replacement

$$L \to L' = L + f(t)$$

   with f an arbitrary function of one variable, does not change the equations of motion for the system.

   (c) Compute the energy associated to the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \cos(q_1 + q_2)\,.$$

   (d) Show that the transformation

$$q_1 \to q_1 + \epsilon$$
$$q_2 \to q_2 - \epsilon$$

   with ϵ constant, is a symmetry of the Lagrangian given above.

3. Two beads of unit mass move along a straight horizontal wire, without friction, with positions x_1 and x_2 (we will assume $x_2 > x_1$). They are joined by a spring of natural length a and constant $\kappa = 1$.

   (a) Introduce the generalised coordinates $q_1 = x_1$ and $q_2 = x_2 - a$. Show that the Lagrangian describing the system in these coordinates is

   $$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_2 - q_1)^2 \, .$$

   (b) Write the Euler-Lagrange equations of motion for q_1 and q_2 coming from this Lagrangian.

   (c) Find the normal modes of the system, including possible zero modes, and write the general solution $q_1(t)$ and $q_2(t)$ for the motion of the system in terms of these normal modes.

   (d) Assume that the system starts at rest, with $q_2 = -q_1 = d > 0$. Find the subsequent motion of the system, that is, give explicit expressions for $q_1(t)$ and $q_2(t)$ compatible with the given initial conditions.

4. Suppose that $\phi_j(x)$ form an orthonormal basis of Hamiltonian eigenfunctions with distinct eigenvalues E_j.

   (a) A particle has normalised wavefunction $\psi(x)$. Using the inner product $\langle \cdot, \cdot \rangle$, write down an expression for the probability P_j to measure energy E_j .

   (b) Show that the wavefunction

   $$\psi(x) = \sum_j c_j \, \phi_j(x)$$

   is correctly normalised if and only if $\sum_j |c_j|^2 = 1$.

   (c) Find the probabilities P_j in terms of the c_j and show that $\sum_j P_j = 1$.

   (d) Now suppose that

   $$\psi(x) = C\left(\phi_1(x) + 2\phi_2(x) + 3\phi_3(x)\right) \, .$$

   Find the normalisation C and probabilities P_1, P_2, P_3.

   (e) Suppose a measurement of energy yields the result E_1. What is the wavefunction immediately after the measurement?

5. Consider a particle of mass m in a potential $V(x)$.

   (a) What is Schrödinger's equation for the wavefunction $\psi(x,t)$?

   (b) Write down an expression for the probability density $P(x,t)$ and show that

   $$\partial_t P(x,t) = -\partial_x J(x,t)$$

   where you should determine $J(x,t)$.

   (c) Write down a definite integral for the probability $P_{ab}(t)$ to find the particle in the region $a < x < b$ and show that

   $$\frac{d}{dt} P_{ab}(t) = J(a,t) - J(b,t)\,.$$

   What is the physical interpretation of this result?

6. A particle confined to $x > 0$ has wavefunction

   $$\psi(x) = Cxe^{-x/a}$$

   where $a > 0$ is a constant.

   (a) Determine the normalisation C.

   (b) Sketch the probability density. Where is the particle most likely to be found?

   (c) Show that $\psi(x)$ is a Hamiltonian eigenfunction for a particle of mass m in a potential of the form

   $$V(x) = -\frac{A}{x}\,.$$

   Determine the constant A and the energy eigenvalue.

SECTION B

7. We describe small oscillations of an infinite one-dimensional string by a one-dimensional field $u(x,t)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2$$

where $u_x \equiv \partial u/\partial x$ and $u_t \equiv \partial u/\partial t$.

(a) Derive the wave equation describing the dynamics of the string.

(b) State D'Alembert's general solution of the wave equation you just found, and show that it is indeed a solution.

(c) We deform the string into a Gaussian shape at $t = 0$, and release it from rest, namely

$$u(x,0) = e^{-x^2} \qquad \text{and} \qquad u_t(x,0) = 0\,.$$

Find the solution for the motion of the string compatible with these initial conditions.

(d) Define the energy contained on an interval $a \le x \le b$ to be

$$E(a,b) = \int_a^b dx\, \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right)\,.$$

Show, using the wave equation you found above, that

$$\frac{dE(a,b)}{dt} = \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_a^b\,.$$

(e) Assume that we attach a mass m to the string, fixed at $x = 0$, so that there is an extra finite contribution to the kinetic energy of the string coming from the mass at $x = 0$. (As above, we ignore the effect of gravity.) Taking a monochromatic wave ansatz of the form

$$u(x,t) = \begin{cases} \Re\left((e^{i\omega x} + Re^{-i\omega x})e^{-i\omega t}\right) & \text{for } x < 0 \\ \Re\left(Te^{i\omega x}e^{-i\omega t}\right) & \text{for } x > 0 \end{cases}$$

where $\Re(f)$ denotes taking the real part of f, solve for R and T in the presence of the mass. [*Hint: use energy conservation at $x = 0$.*]

8. Consider an infinitesimal transformation of the generalised coordinates of the form

$$q_i \to q_i' = q_i + \epsilon a_i(q_1, \ldots, q_n) \qquad ; \qquad \dot{q}_i \to \dot{q}_i' = \dot{q}_i + \epsilon \dot{a}_i(q_1, \ldots, q_n)$$

where we have dropped possible terms of quadratic and higher order in the infinitesimal parameter ϵ.

(a) Show, using the Euler-Lagrange equations, that under such a transformation the change in the Lagrangian

$$\delta L = L' - L = L(q_1', \ldots, q_n', \dot{q}_1', \ldots, \dot{q}_n') - L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

is given by

$$\delta L = \epsilon \frac{dQ}{dt} + O(\epsilon^2)$$

for some Q that you should find explicitly. In the special case $\delta L = O(\epsilon^2)$ the transformation is a symmetry, and the Q that you found is a conserved charge.

(b) Consider a Lagrangian of the form

$$L_{ab} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - a q_1^2 - b q_2^2 \,.$$

Find the relation that needs to be satisfied between a and b so that the infinitesimal rotation

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

is a symmetry of L_{ab}.

(c) Denote by L_a the Lagrangian with a arbitrary, but b chosen so that the rotation above is a symmetry. Compute the conserved charge Q associated to the rotation of L_a.

(d) Compute the generalised momenta p_i for the Lagrangian L_{ab}, and check that the variation of q_i under the transformation generated by Q, obtained by computing the relevant Poisson bracket, agrees with the transformation you started with.

(e) Write down the expression for the Hamiltonian H_{ab} in terms of q_i and p_i. Then show, by computing the Poisson bracket $\dot{Q} = \{Q, H_{ab}\}$, that Q is conserved if and only if the relation you found between a and b above holds.

9. Consider a particle of mass m in the semi-infinite potential well

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le L \\ V_0 & x > L \end{cases} ,$$

where $0 < V_0 < \infty$.

(a) Explain why the wavefunction should vanish for $x < 0$. Find constants $k > 0$ and $\kappa > 0$ in terms of E, V_0, $\hbar$ and m such that

$$\psi(x) = \begin{cases} A\cos(kx) + B\sin(kx) & 0 \le x \le L \\ Ce^{-\kappa x} & x > L \end{cases}$$

is a Hamiltonian eigenfunction with energy $0 < E < V_0$.

(b) What are the boundary conditions at $x = 0$ and $x = L$?

(c) Thus determine A and eliminate B and C to derive the quantization condition

$$-\cot(z) = \sqrt{z_0^2/z^2 - 1}\,,$$

where

$$z = kL \qquad z_0 = \sqrt{k^2 + \kappa^2}\,L$$

are dimensionless parameters.

(d) Illustrate the solutions of the quantisation condition by sketching the functions $-\cot(z)$ and $\sqrt{z_0^2/z^2 - 1}$ on the same graph.

(e) Hence show that in the limit $V_0 \to \infty$ there are an infinite number of solutions with energies

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 , \quad n \in \mathbb{Z}_{>0}\,.$$

10. The Hamiltonian operator of a simple harmonic oscillator with mass m and angular frequency ω is

$$\hat{H} = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) ,$$

where

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) .$$

(a) Using the canonical commutator $[\hat{x}, \hat{p}] = i\hbar$, show that $[a, a^\dagger] = 1$ and hence prove by induction that

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} , \quad n \geq 1 .$$

(b) The ground state wavefunction is defined by $a\,\phi_0(x) = 0$. Show that

$$\phi_n(x) = \frac{1}{\sqrt{n!}}(a^\dagger)^n \phi_0(x)$$

are Hamiltonian eigenfunctions for all $n \geq 0$ and determine their energies.

(c) Now write down expressions for a and $a^\dagger$ as differential operators.

(d) Show that the ground state wavefunction has the form

$$\phi_0(x) = Ce^{-\alpha x^2}$$

and determine the constant α and normalisation C. Hence compute the first excited wavefunction $\phi_1(x)$.

(e) Sketch both of the wavefunctions $\phi_0(x)$, $\phi_1(x)$.

(f) Now consider the "half" simple harmonic oscillator with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ \frac{1}{2}m\omega^2 x^2 & x \geq 0 . \end{cases}$$

Write down the boundary condition at $x = 0$ and explain why

$$\psi(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \phi_1(x) & \text{if} \quad x \geq 0 \end{cases}$$

is a Hamiltonian eigenfunction in this potential.

You may use without proof the definite integral $\int_{-\infty}^{\infty} e^{-y^2} = \sqrt{\pi}$.