EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2581-WE01

Title:
Algebra II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let r be a non-zero element of an integral domain R. Show that the function $f\colon R \to R$ given by $f(x) = rx$ is injective.
   (b) Deduce that any finite integral domain is a field.
   (c) Is the map f from part (a) always a ring homomorphism? If yes, provide a proof, otherwise give a counterexample.

2. (a) Define what it means to be a unit in a ring.
   (b) Show that the set of all units $R^\times$ in a ring R forms a group under multiplication.
   (c) Let $R := \mathbb{Z}[\sqrt{17}] = \{a + b\sqrt{17} \mid a, b \in \mathbb{Z}\}$, a subring of $\mathbb{C}$. Show that the principal ideal $I = (4 - \sqrt{17})$ is the full ring R.

3. Factorize the following polynomials into irreducible factors.
   (a) $f(x) = x^3 - x + \bar{4} \in (\mathbb{Z}/7)[x]$.
   (b) $f(x) = x^5 + 2x^4 + 3x^3 + 3x^2 - 6x - 3 \in \mathbb{Q}[x]$.
   (c) $f(x) = x^8 - 8x^4 + 16 \in \mathbb{Q}[x]$.

4. (a) State the Burnside lemma for counting the number of orbits of the action of a finite group G on a finite set X.
   (b) Compute the number of essentially different ways to colour in the faces of a regular tetrahedron red, white and blue.

5. Consider the element $\sigma := (13)(5321)(42) \in S_5$.
   (a) Compute σ^{100}. Justify your computation carefully.
   (b) How many elements of S_5 are conjugate to σ?

6. For each of the following pairs of groups, are they isomorphic? Prove your assertion, stating your reasons clearly.
   (a) The direct product $C_3 \times C_3$ of the cyclic group of order 3 with itself and the cyclic group C_9 of order 9.
   (b) The quaternion group Q_8 and the dihedral group D_8, both of order 8.
   (c) The multiplicative groups $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \times, 1)$ and $\mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \times, 1)$.

SECTION B

7. Let p be a fixed prime number and let $R = \{\frac{a}{b} \in \mathbb{Q} \mid \gcd(b, p) = 1\}$.

   (a) Show that R is an integral domain.

   (b) Show that the map $\varphi \colon R \to \mathbb{Z}/p^n$ given by $\frac{a}{b} \mapsto \overline{a}\overline{b}^{-1}$ is well-defined (i.e. does not depend on the way to write a/b in R) and that φ is a ring homomorphism.

   (c) Show that φ is surjective and express the kernel of φ as a principal ideal in R. Give careful reasoning for each step.

   (d) Express $\mathbb{Z}/p^n$ as a quotient ring of R.

8. (a) State and prove Eisenstein's criterion for irreducibility.
   (Hint: You may reduce modulo a suitable prime.)

   (b) Let $R = \mathbb{Z}[\sqrt{-13}]$.

      i. Considering decompositions of 14 or otherwise, show that 2 is irreducible in R but not prime.
      (Hint: You may use that the norm function $N(a + b\sqrt{-13}) = a^2 + 13b^2$ is multiplicative.)

      ii. Prove that the principal ideal $(2) \subseteq R$ is not maximal.

9. (a) If a finite group G has even order show there exists an element of order two.

   (b) Let G be a group with $\mathrm{ord}(g) = 2$ for all $g \neq e$. Show that G is abelian. *(Hint: consider $ghg^{-1}h^{-1}$.)*

   (c) Let G be an abelian group. Show that $\{g \in G \mid \mathrm{ord}(g) < \infty\}$ is a subgroup.

   (d) Let H and K be normal subgroups of a finite group G with $H \cap K = \{e\}$ and $|H| \cdot |K| = |G|$. Show that $G \cong H \times K$. *(Hint: prove that $(h, k) \mapsto hk$ is a bijective homomorphism from $H \times K$ to G. You may need to show that $hkh^{-1}k^{-1} = e$.)*

10. You may use Q9 part (d) in this question.

   (a) State the Sylow theorem.

   (b) Suppose that m and n are coprime. Prove that $C_n \times C_m \cong C_{nm}$.

   (c) Prove that every group of order 217 is cyclic. *(Hint: what are the prime factors of 217? Show that G is a direct product of two Sylow subgroups.)*