Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2627-WE01

Title:
Geometric Topology II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

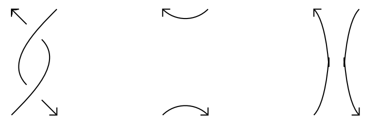
ED01/2019

SECTION A

1. (a) State the defining relations of the Alexander-Conway polynomial.

   (b) Calculate the Alexander-Conway polynomial of the Hopf link with positive linking number.

   (c) Let L be an oriented link diagram containing a tangle as in Figure 1(a), and let L_{00} be the link diagram obtained from L by replacing this tangle with the tangle from Figure 1(b), and L_∞ the link diagram obtained from L by replacing this tangle with the tangle from Figure 1(c).



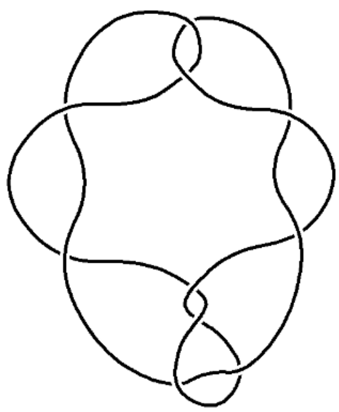
(a) (b) (c)

Figure 1.

   Derive a formula relating the Alexander-Conway polynomials of L, L_{00} and L_∞.

   (d) Show that the trefoil knot admits a diagram only involving tangles as in Figure 1(a) (possibly with different orientations), and calculate its Alexander-Conway polynomial.

2. (a) Apply Seifert's algorithm to the knot diagram below.

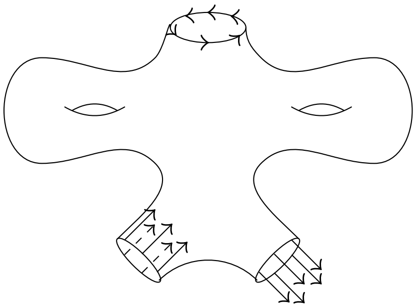


   (b) Determine the genus of the resulting surface.

   (c) Calculate the genus of this knot by first simplifying it.

3. (a) State the Poincaré-Hopf Theorem.

   (b) Assume the following surface has a vector field with exactly one singularity on the interior, and where the vector field is as indicated on the boundary. Determine the index of the singularity.

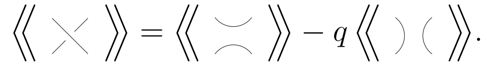


SECTION B

4. Given a link diagram D, define the *double bracket polynomial* of D, denoted $\langle\langle D\rangle\rangle$, as the Laurent polynomial in q obtained by the following rules.

   (DP1) $\langle\langle O\rangle\rangle = q + q^{-1}$, where O is the standard diagram of the unknot.

   (DP2) $\langle\langle D_1 \sqcup D_2\rangle\rangle = \langle\langle D_1\rangle\rangle \cdot \langle\langle D_2\rangle\rangle$, where $D_1 \sqcup D_2$ is the disjoint union of link diagrams D_1 and D_2.

   (DP3) 

   (a) Calculate $\langle\langle D\rangle\rangle$ for D a diagram of

   - the unlink with three components and no crossings.
   - the unknot with exactly one positive crossing and no negative crossings.
   - the unknot with exactly one negative crossing and no positive crossings.
   - the Hopf link with a two crossing diagram. You only need to do one of the two possible diagrams.
   - the right-handed trefoil knot with exactly three positive crossings and no negative crossings.

   (b) For an oriented link diagram D denote by $n^+(D)$ the number of positive crossings, and denote by $n^-(D)$ the number of negative crossings in the diagram D. Let
   $$K(D) = (-1)^{n^-(D)} q^{n^+(D)-2n^-(D)} \langle\langle D\rangle\rangle.$$
   Show that $K(D)$ is invariant under the three Reidemeister moves.

   (c) Calculate $K(L)$ for L the Hopf link with linking number -1, and $K(T)$ for T the right-hand trefoil knot.

5. (a) State the defining relations of the absolute polynomial $Q_L(x)$ of a link L.

   (b) Determine the absolute polynomial of the unlink U_n with n components for each $n \geq 2$.

   (c) Let L_1 and L_2 be links. Show that
   $$Q_{L_1 \sqcup L_2}(x) = Q_{U_2}(x) \cdot Q_{L_1}(x) \cdot Q_{L_2}(x),$$
   where $L_1 \sqcup L_2$ is the link obtained by disjoint union.

   (d) Let K_1 and K_2 be knots with K_2 invertible. Show that
   $$Q_{K_1+K_2}(x) = Q_{K_1}(x) \cdot Q_{K_2}(x),$$
   where $K_1 + K_2$ is the composition of the two knots.

6. (a) State the definition of the index $I_v(p)$ of an isolated singularity p of a vector field v.

(b) Draw the tangent curves of a vector field v in the plane which has an isolated singularity

- of index 4.
- of index -2.

(c) Let $v\colon \mathbb{C} \to \mathbb{C}$ be a vector field in the plane with an isolated singularity p. Let w be the vector field given by

$$w(z) = \overline{v(z)},$$

the complex conjugate of v. Show that p is an isolated singularity of w, and express $I_w(p)$ in terms of $I_v(p)$.

(d) Let $v\colon \mathbb{C} \to \mathbb{C}$ be a vector field in the plane with an isolated singularity p. Let u be the vector field given by

$$u(z) = v(z) \cdot v(z),$$

the square of v. Show that p is an isolated singularity of u, and express $I_u(p)$ in terms of $I_v(p)$.