

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH2647-WE01

Title:

Probability II

Time Allowed:	2 hours					
Additional Material provided:	None					
Materials Permitted:	None					
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.				
Visiting Students may use dictionaries: No						

Instructions to Candidates:	Credit will be given for the best TWO and the best TWO answers from Sec Questions in Section B carry ONE an marks as those in Section A.	tion B.	
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SECTION A

1. Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- (a) Draw the directed graph associated to this Markov chain and find the communicating classes of this Markov chain.
- (b) Which of these communicating classes are closed? Are there any absorbing states?
- (c) Compute $\mathbb{P}[X_n = 5 | X_0 = 5]$ and $\mathbb{P}[X_n = 5 | X_0 = 4]$ for all $n \ge 1$.
- (d) Which states of this Markov chain are transient and which states are recurrent? Carefully state any theorems you use in determining the classification of the states.
- 2. (a) Suppose that X and Y are independent random variables having the Poisson distribution with parameter α and β respectively, that is, for $n \ge 0$

$$\mathbb{P}[X=n] = \frac{\alpha^n e^{-\alpha}}{n!}$$
 and $\mathbb{P}[Y=n] = \frac{\beta^n e^{-\beta}}{n!}$.

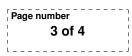
- (i) Find the probability generating functions of X and Y.
- (ii) Find the probability generating function of X+Y. What is the distribution of X+Y?
- (b) (i) Suppose that Z is a random variable taking non-negative integer values with

$$\mathbb{P}[Z=n] = \begin{cases} \frac{1}{m+1} & \text{for } 0 \le n \le m\\ 0 & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{N}$ is a constant. Show that the probability generating function of Z is given by

$$\frac{1-s^{m+1}}{(1+m)(1-s)}.$$

(ii) A player can score 0, 1 or 2 points in a game with probabilities $\frac{1}{10}$, $\frac{6}{10}$ and $\frac{3}{10}$ respectively. A sequence of N independent games are played where N is the value from rolling a fair dice (which takes values 1 through to 6). Find the probability generating function of the total sum of scores obtained and find the expected total sum.



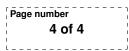


- 3. (a) Carefully state the monotone convergence theorem.
 - (b) Let X and X_1, X_2, \ldots be random variables such that for $n \ge 1, X_n \ge X_{n+1} \ge 0$ holds almost surely and $\lim_{n\to\infty} X_n = X$ almost surely. Suppose that $\mathbb{E}[X_1] < \infty$. Prove that

$$\lim_{n \to \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Hint: Consider $Y_n = X_1 - X_n$.

- (c) Consider $\Omega = [0, 1]$ with probability measure given by the uniform distribution, that is $\mathbb{P}[\omega \in [a, b]] = b a$ for any $0 \le a \le b \le 1$. Define $Y_n(\omega) = 1/(n\omega)$ for $\omega \in (0, 1]$ and $Y_n(0) = 1$ for $n \ge 1$.
 - (i) Show that Y_n converges to 0 almost surely and that $Y_n(\omega) \ge Y_{n+1}(\omega)$ for all $\omega \in (0, 1]$ and $n \ge 1$.
 - (ii) Show that $\mathbb{E}[Y_n] = \infty$ for all $n \ge 1$.
 - (iii) Does $\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[\lim_{n\to\infty} Y_n]$? If not, why does the result in (b) not apply?



SECTION B

- 4. Let X and X_1, X_2, \ldots be random variables.
 - (a) Give complete definitions of $\lim_{n\to\infty} X_n = X$ in probability, $\lim_{n\to\infty} X_n = X$ almost surely and $\lim_{n\to\infty} X_n = X$ in L^2 .
 - (b) Show that if $\lim_{n\to\infty} X_n = X$ in probability and $\lim_{n\to\infty} Y_n = Y$ in probability, then $\lim_{n\to\infty} X_n + Y_n = X + Y$ in probability.
 - (c) In the questions below, justify your answer by either proving the result or giving a counterexample. Carefully state any results you use.
 - (i) Does $\lim_{n\to\infty} X_n = X$ in L^2 imply that $\lim_{n\to\infty} X_n = X$ in probability?
 - (ii) Does $\lim_{n\to\infty} X_n = X$ in probability imply that $\lim_{n\to\infty} X_n = X$ in L^2 ?
 - (iii) Suppose that there exists $M \ge 0$ such that $\mathbb{P}[|X_n| < M] = 1$ for all $n \ge 1$ and that $\lim_{n\to\infty} X_n = 0$ in probability. Does $\lim_{n\to\infty} X_n = 0$ in L^2 .
- 5. (a) Let $(A_n)_{n\geq 1}$ be a sequence of events such that $A_n \subset A_{n+1}$ for all $n \geq 1$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Show that

$$\mathbb{P}[A] = \lim_{n \to \infty} \mathbb{P}[A_n].$$

- (b) Suppose that X is a non-negative random variable. Prove the following results:
 - (i) if $\mathbb{E}[X] < \infty$, then $\mathbb{P}[X < \infty] = 1$,
 - (ii) if $\mathbb{E}[X] = 0$, then $\mathbb{P}[X = 0] = 1$.
- (c) State the Borel Cantelli Lemmas making sure to define any notation you use.
- (d) Let $(B_n)_{n\geq 1}$ be a sequence of independent events such that $\mathbb{P}[B_n] = 1/n$ for $n \geq 1$. Find $\mathbb{P}[\limsup B_n]$.
- 6. (a) (i) A flea hops randomly on the vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.
 - (ii) Another flea hops randomly on vertices of a triangle, but this time, it is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops the flea is back where it started? Hint: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}}e^{\pm i\pi/6}$.
 - (b) Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, 2, ...\}$ with transition probabilities given by

$$\mathbb{P}[X_n = 1 | X_{n-1} = 0] = 1, \qquad \mathbb{P}[X_n = k + 1 | X_{n-1} = k] = p_k, \\ \mathbb{P}[X_n = k - 1 | X_{n-1} = k] = 1 - p_k = q_k, \qquad \text{and } 0 \le p_k \le 1$$

for all $n, k \ge 1$. Let, for $k \ge 1$

$$\gamma_k = \frac{q_k q_{k-1} \dots q_1}{p_k p_{k-1} \dots p_1}.$$

Assume that $\sum_{k=1}^{\infty} \gamma_k < \infty$ and suppose that $X_0 = 0$. Find a formula in terms of $(\gamma_k)_{k\geq 1}$ for the probability that $X_n \geq 1$ for all $n \geq 1$. Hint: Try using $u_i = h_{i-1} - h_i$ where h_i is appropriately defined.