Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2647-WE01

Title:
Probability II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Let $(X_n)_{n\geq 0}$ be a (time-homogeneous) Markov chain with state space $S = \{1,2,3,4,5\}$ and transition matrix given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw the directed graph associated to this Markov chain and find the communicating classes of this Markov chain.

(b) Which of these communicating classes are closed? Are there any absorbing states?

(c) Compute $\mathbb{P}[X_n = 5|X_0 = 5]$ and $\mathbb{P}[X_n = 5|X_0 = 4]$ for all $n \geq 1$.

(d) Which states of this Markov chain are transient and which states are recurrent? Carefully state any theorems you use in determining the classification of the states.

2. (a) Suppose that X and Y are independent random variables having the Poisson distribution with parameter α and β respectively, that is, for $n \geq 0$

$$\mathbb{P}[X = n] = \frac{\alpha^n e^{-\alpha}}{n!} \quad \text{and} \quad \mathbb{P}[Y = n] = \frac{\beta^n e^{-\beta}}{n!}.$$

(i) Find the probability generating functions of X and Y.

(ii) Find the probability generating function of $X+Y$. What is the distribution of $X+Y$?

(b) (i) Suppose that Z is a random variable taking non-negative integer values with

$$\mathbb{P}[Z = n] = \begin{cases} \frac{1}{m+1} & \text{for } 0 \leq n \leq m \\ 0 & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{N}$ is a constant. Show that the probability generating function of Z is given by

$$\frac{1 - s^{m+1}}{(1+m)(1-s)}.$$

(ii) A player can score 0, 1 or 2 points in a game with probabilities $\frac{1}{10}$, $\frac{6}{10}$ and $\frac{3}{10}$ respectively. A sequence of N independent games are played where N is the value from rolling a fair dice (which takes values 1 through to 6). Find the probability generating function of the total sum of scores obtained and find the expected total sum.

3. (a) Carefully state the monotone convergence theorem.

   (b) Let X and $X_1, X_2, \ldots$ be random variables such that for $n \geq 1$, $X_n \geq X_{n+1} \geq 0$ holds almost surely and $\lim_{n\to\infty} X_n = X$ almost surely. Suppose that $\mathbb{E}[X_1] < \infty$. Prove that

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

   Hint: Consider $Y_n = X_1 - X_n$.

   (c) Consider $\Omega = [0, 1]$ with probability measure given by the uniform distribution, that is $\mathbb{P}[\omega \in [a, b]] = b - a$ for any $0 \leq a \leq b \leq 1$. Define $Y_n(\omega) = 1/(n\omega)$ for $\omega \in (0, 1]$ and $Y_n(0) = 1$ for $n \geq 1$.

   (i) Show that Y_n converges to 0 almost surely and that $Y_n(\omega) \geq Y_{n+1}(\omega)$ for all $\omega \in (0, 1]$ and $n \geq 1$.

   (ii) Show that $\mathbb{E}[Y_n] = \infty$ for all $n \geq 1$.

   (iii) Does $\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[\lim_{n\to\infty} Y_n]$? If not, why does the result in (b) not apply?

SECTION B

4. Let X and $X_1, X_2, \ldots$ be random variables.

   (a) Give complete definitions of $\lim_{n\to\infty} X_n = X$ in probability, $\lim_{n\to\infty} X_n = X$ almost surely and $\lim_{n\to\infty} X_n = X$ in L^2.

   (b) Show that if $\lim_{n\to\infty} X_n = X$ in probability and $\lim_{n\to\infty} Y_n = Y$ in probability, then $\lim_{n\to\infty} X_n + Y_n = X + Y$ in probability.

   (c) In the questions below, justify your answer by either proving the result or giving a counterexample. Carefully state any results you use.

   (i) Does $\lim_{n\to\infty} X_n = X$ in L^2 imply that $\lim_{n\to\infty} X_n = X$ in probability?

   (ii) Does $\lim_{n\to\infty} X_n = X$ in probability imply that $\lim_{n\to\infty} X_n = X$ in L^2?

   (iii) Suppose that there exists $M \geq 0$ such that $\mathbb{P}[|X_n| < M] = 1$ for all $n \geq 1$ and that $\lim_{n\to\infty} X_n = 0$ in probability. Does $\lim_{n\to\infty} X_n = 0$ in L^2.

5. (a) Let $(A_n)_{n\geq 1}$ be a sequence of events such that $A_n \subset A_{n+1}$ for all $n \geq 1$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Show that

$$\mathbb{P}[A] = \lim_{n\to\infty} \mathbb{P}[A_n].$$

   (b) Suppose that X is a non-negative random variable. Prove the following results:

   (i) if $\mathbb{E}[X] < \infty$, then $\mathbb{P}[X < \infty] = 1$,

   (ii) if $\mathbb{E}[X] = 0$, then $\mathbb{P}[X = 0] = 1$.

   (c) State the Borel Cantelli Lemmas making sure to define any notation you use.

   (d) Let $(B_n)_{n\geq 1}$ be a sequence of independent events such that $\mathbb{P}[B_n] = 1/n$ for $n \geq 1$. Find $\mathbb{P}[\limsup B_n]$.

6. (a) (i) A flea hops randomly on the vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.

   (ii) Another flea hops randomly on vertices of a triangle, but this time, it is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops the flea is back where it started?
   Hint: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.

   (b) Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities given by

$$\mathbb{P}[X_n = 1 | X_{n-1} = 0] = 1, \qquad \mathbb{P}[X_n = k+1 | X_{n-1} = k] = p_k,$$
$$\mathbb{P}[X_n = k-1 | X_{n-1} = k] = 1 - p_k = q_k, \qquad \text{and } 0 \leq p_k \leq 1$$

   for all $n, k \geq 1$. Let, for $k \geq 1$

$$\gamma_k = \frac{q_k q_{k-1} \cdots q_1}{p_k p_{k-1} \cdots p_1}.$$

   Assume that $\sum_{k=1}^{\infty} \gamma_k < \infty$ and suppose that $X_0 = 0$. Find a formula in terms of $(\gamma_k)_{k\geq 1}$ for the probability that $X_n \geq 1$ for all $n \geq 1$.
   Hint: Try using $u_i = h_{i-1} - h_i$ where h_i is appropriately defined.