Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2657-WE01

Title:
Special Relativity & Electromagnetism II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Find the charge density ρ and the electrostatic potential φ associated with the electric field with components

$$E_x = -2xyz\, e^{-x^2}\,, \quad E_y = z\, e^{-x^2}\,, \quad E_z = y\, e^{-x^2}\,.$$

(b) Find the current density $\mathbf{J}$ associated with the magnetic field

$$B_x = yz\,, \quad B_y = xz\,, \quad B_z = 0\,,$$

and check that $\operatorname{div}\mathbf{J} = 0$ for your result.

2. (a) Let (r, θ, z) be cylindrical polar coordinates with $r = \sqrt{x^2 + y^2}$. Use Ampère's law to find the magnetic field $\mathbf{B}$ due to a current density $\mathbf{J} = \mathbf{e}_z\, e^{-r}/r$. You may assume that because of cylindrical symmetry $\mathbf{B}$ points in the direction of $\mathbf{e}_\theta = -y\mathbf{e}_x/r + x\mathbf{e}_y/r$ and has a magnitude that only depends on r.

(b) An infinitely long, but infinitesimally thin, amber rod lies along the y-axis. The rod is electrostatically charged with a uniform charge density σ per unit length of the rod. Assuming that the electric field points in the direction of the vector $\mathbf{V} = x\mathbf{e}_x + z\mathbf{e}_z$ and has a magnitude that depends only on the distance from the y-axis use Gauss' law to find the magnitude and direction of this field at the point (x, y) in the xy-plane.

3. (a) Find the interval between two events, one with co-ordinates $(8, 1, 3, 4)$, the other with co-ordinates $(3, 0, 1, 2)$. Is this space-like or time-like? If it is space-like find the spatial distance between the events in a frame in which they occur at the same time. If it is time-like find the time between the events in a frame in which they occur at the same spatial point.

(b) Find the speed in an inertial frame at which a rod must move in the direction of its axis for its length to be measured in the same inertial frame to be one-third of its length at rest. An observer is at rest in the inertial frame at a point on the axis of the rod and watches the rod approach. A flashlight tied to the front of the rod points in the direction of motion and emits a light of frequency ν. What is the ratio of the observed frequency ν' to ν?

SECTION B

4. Maxwell's equations in free space are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \quad \nabla \cdot \mathbf{B} = 0\,,$$

$$\nabla \cdot \mathbf{E} = 0\,, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\,.$$

(a) Given that

$$\mathbf{E} = \sin\left(k\left(x - ct\right)\right)\, \mathbf{e}_y$$

with constants k and c satisfies these equations, integrate the first of Maxwell's equations with respect to time to find a solution for $\mathbf{B}$. (You may ignore the integration constant.) Use the last of Maxwell's equations to relate c to ϵ_0 and μ_0.

(b) Use Maxwell's equations to relate the divergence of $\mathbf{E}\times\mathbf{B}$ to the time derivative of $\epsilon_0\,\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}/\mu_0$ for electromagnetic fields in free space. (You may find the identity $\nabla \cdot (\mathbf{U} \times \mathbf{W}) = (\nabla \times \mathbf{U}) \cdot \mathbf{W} - \mathbf{U} \cdot \nabla \times \mathbf{W}$ useful). Hence show that for any volume V (that does not vary in time) with boundary surface S

$$\frac{d}{dt} \int_V \frac{1}{2}\left(\epsilon_0\, \mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}/\mu_0\right) = \alpha \oint_S \mathbf{E} \times \mathbf{B} \cdot d\mathbf{A}\,,$$

and find the coefficient α.

(c) Verify this relation for the electromagnetic field you found in part (a) by explicitly calculating both sides in the case that V is the cube with $|x| < 1/2$, $|y| < 1/2$, and $|z| < 1/2$.

5. (a) The interval between two events with co-ordinates x_A^μ and x_B^μ is defined by

$$s(A,B)^2 = \eta_{\mu\nu}\,(x_B^\mu - x_A^\mu)(x_B^\nu - x_A^\nu)\,.$$

Derive the condition on the quantities $L^\mu{}_\nu$ for the transformation expressed in component form as

$$x^\mu \to x'^\mu = L^\mu{}_\nu\, x^\nu$$

to leave $s(A,B)^2$ invariant. Show that this is satisfied by taking $L^\mu{}_\nu = \delta^\mu{}_\nu$. Also show that if $L_1^\mu{}_\nu$ and $L_2^\mu{}_\nu$ both satisfy this relation then so does $L_1^\mu{}_\rho\, L_2^\rho{}_\nu$.

(b) A tensor is a generalisation of a vector. Under Lorentz transformations the components $H^{\mu\nu}$ of the tensor H transform as

$$H^{\mu\nu} \to H'^{\mu\nu} = L^\mu{}_\rho\, L^\nu{}_\sigma\, H^{\rho\sigma}$$

Use the chain rule to show that if H depends on position in spacetime then

$$J^\nu \equiv \frac{\partial H^{\mu\nu}}{\partial x^\mu}$$

transforms as the components of a vector under Lorentz transformations.

(c) If a four vector has components V^μ what conditions must it satisfy to be (i) space-like, (ii) time-like, (iii) light-like? Show that the sum of two non-zero, future-pointing, light-like vectors is light-like if their components are proportional. If their components are not proportional is the sum of two light-like vectors space-like or time-like? Justify your answer.

6. A particle in the upper atmosphere is at rest in the frame of a laboratory at sea-level. It is observed to decay to produce two identical particles each of which has a rest mass equal to one-quarter of the rest mass of the original particle. One of these travels vertically downwards through one kilometre of atmosphere (as measured in the laboratory frame) to decay in a detector in the laboratory where it is identified as a muon. In this question express your answers in terms of the speed of light denoted by c, do not use its numerical value.

(a) Use conservation of four momentum to find the speeds and direction of motion of the two particles in the laboratory frame. What is the speed of either muon in the rest-frame of the other?

(b) What is the lifetime in its own rest-frame of the muon that decays in the detector? What is the lifetime of the muon that decays in the detector when measured in the rest-frame of the other muon?

(c) If the first particle had decayed from rest into three identical muons each moving with the same speed in the laboratory frame and one of them is travelling vertically downwards how high above the laboratory does it decay (assuming its lifetime in its own rest frame is the same as before)?