Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH2667-WE01

Title:
Monte Carlo II

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	Tables: Normal.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **ONE** answer from Section B. Questions in Section B carry **THREE TIMES** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) State the algorithm for the *Inverse Transform Method* for sampling from a random variable X with cumulative distribution function F_X.

   (b) Let X be a Geometric random variable with parameter p and support $\{0, 1, 2, \ldots\}$. The probability mass function of X is given by

$$P(X = x) = \begin{cases} p(1-p)^x, & x = 0, 1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

   and its cumulative distribution function is given by

$$F_X(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1 - (1-p)^{\lfloor x+1 \rfloor}, & x \geq 0 \end{cases}$$

   where $\lfloor \cdot \rfloor$ is the floor function. Write the procedure for sampling from X using the *Inverse Transform Method*.

   (c) Assume you have sampled from the $U(0, 1)$ (uniform distribution on $(0, 1)$) and obtained the following values: $u_1 = 0.1$, $u_2 = 0.6$, and $u_3 = 0.8$. Use the procedure outlined in item (b) and the samples u_1, u_2, u_3 provided to generate 3 samples from the Geometric distribution with $p = 0.5$.

2. (a) State the definition of a Copula.

   (b) Prove that if X is a continuous random variable with cumulative distribution function F_X, then the Probability Integral Transform $F_X(X)$ is uniformly distributed on $(0, 1)$, that is, $F_X(X) \sim U(0, 1)$.

   (c) Outline an algorithm for generating a correlated sample from two random variables Y and Z with cumulative distribution functions F_Y and F_Z, respectively, using a Gaussian Copula.

3. (a) Let X be a Laplace (or Double Exponential) random variable with probability density function given by

$$f_X(x) = \frac{1}{2} \exp\left(-|x|\right), \quad x \in \mathbb{R}.$$

   Calculate the cumulative distribution function F_X of X.

   (b) Write the procedure for sampling from X using F_X using the Inverse Transform Method.

   (c) Use the Composition Method to write an alternative sampling procedure for X.

SECTION B

4. Let X be a random variable with probability density function given by

$$f_X(x) = \begin{cases} ax(1-x^3), & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

(a) Calculate a and compute the cumulative distribution function of X. Comment on whether the Inverse Transformation Method would be adequate to sample from X.

(b) Assume we use a Uniform distribution on $(0,1)$ as a proposal distribution for the Acceptance-Rejection Method to sample from X. Find the unconditional acceptance rate and the acceptance probability function. State the steps to be followed to sample from X.

(c) Estimate the average number of times the first step of the algorithm will be repeated before the algorithm returns a valid sample.

(d) Change the proposal distribution to the triangular distribution with probability density function given by

$$h_Y(x) = \begin{cases} 2-2x, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the unconditional acceptance rate and the acceptance probability function. State the new version of your Acceptance-Rejection algorithm.

(e) Estimate the number of iterations needed to draw a sample from X using the algorithm in (d). Compare to the results in (c).

5. Let U be a random variable with Uniform distribution on $(0,1)$. We are interested in using the integral

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$$

to estimate π.

(a) Show that

$$E(\sqrt{1-U^2}) = \frac{\pi}{4}$$

(b) Show that

$$Var(\sqrt{1-U^2}) = \frac{2}{3} - \left(\frac{\pi}{4}\right)^2$$

(c) Given an independent sample $U_1, \ldots, U_{100}$ from U of size 100, and using (a), write down a Monte Carlo estimator for π, and its sample variance.

(d) Devise a method for estimating π by stratifiying the sample in (c). Write down a new estimator for π using this stratified sample. Comment on the difference between your new estimator and the estimator in (c).

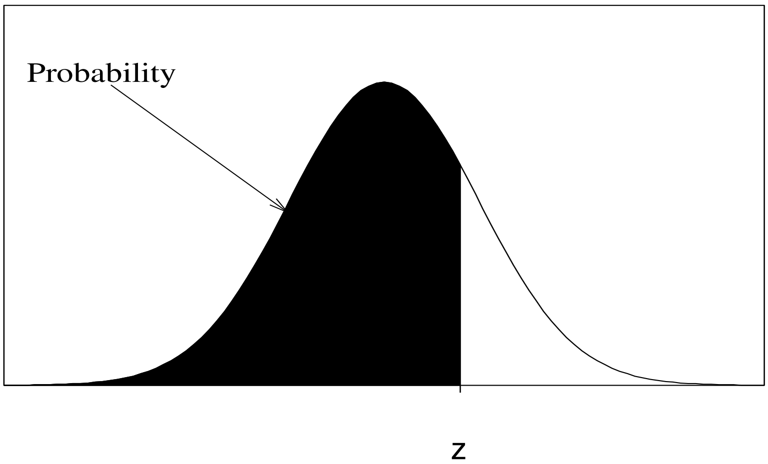
Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z, i.e. $\Phi(z)$.

For $z > 3$,

$$1 - \Phi(z) \approx \frac{1}{\sqrt{2\pi}z} e^{-\frac{1}{2}z^2}$$

is accurate to within 10% of the true value.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998