

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH3021-WE01

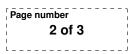
Title:

Differential Geometry III

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

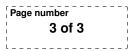
Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section and the best THREE answers from S Questions in Section B carry TWICE in Section A.	ection B.	arks as those
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Revision:



SECTION A

- 1. (a) Let $\alpha : (a, b) \to \mathbb{R}^2$ be a map. Define what it means for α to be a regular parametrized plane curve. Define what it means for α to be parametrized by arclength.
 - (b) Prove or disprove the following claim : " $\alpha(t) = (t^2, 2t^3)$ is a regular parametrized plane curve".
 - (c) Define the curvature $\kappa(t)$ of a regular parametrized plane curve $\alpha(t)$.
- 2. Let $\alpha(t) = (t, t^2, t^3 + 1)$ be a space curve.
 - (a) Compute the curvature of α at p = (0, 0, 1).
 - (b) Compute the unit normal vector N(0) of α at $\alpha(0)$.
- 3. (a) Give the definition of a regular surface $S \subset \mathbb{R}^3$.
 - (b) Let $x : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$, $x(u, v) = (v \cos u, v \sin u, v)$ and let $S \subset \mathbb{R}^3$ be the image of x. Prove that S is a regular surface in \mathbb{R}^3 .
- 4. Let $S \subset \mathbb{R}^3$ be a regular surface and let $x : U \to S$ be a regular parametrization of S.
 - (a) Define the Gauss map on S, the Weingarten map at $p \in S$, and define what it means that $p \in S$ is elliptic/parabolic/hyperbolic.
 - (b) For which $a \ge 0$ are all points of $S = \{x_1^2 + ax_2^2 = 1\} \subset \mathbb{R}^3$ parabolic ? Prove your answer.
- 5. Let $S = \{x_3 = x_1^2 x_1x_2 + 2x_2^2\} \subset \mathbb{R}^3$.
 - (a) Show that S may be represented as $f^{-1}(b)$ for the regular value b of a smooth function f. Give a regular parametrization x(u, v) of S and prove it has the required properties.
 - (b) Compute the Gauss curvature of S at $p = (1, 1, 2) \in S$.
- 6. Consider a regular surface $S \subset \mathbb{R}^3$ parametrised by x = x(u, v).
 - (a) Give the definition of a geodesic on a surface.
 - (b) Let S parametrized by $x(u,v) = (u^2 v^2, v, u)$. Show that $S \cap \{x_1 = 0\}$ contains a geodesic.



SECTION B

- 7. (a) Let $S_1, S_2 \subset \mathbb{R}^3$ be two regular surfaces and $F: S_1 \to S_2$ be a map. Define what it means that F is differentiable at $p \in S_1$ and define $dF(p): T_pS_1 \to T_{F(p)}S_2$.
 - (b) Define what it means for F as in (a) to be an isometry.
 - (c) Let $S_1 = S_2 = \{x_1^2 + x_2^2 x_3 = 0\}$ and $F(x_1, x_2, x_3) = (-x_2, x_1, x_3)$. Prove that and $F: S_1 \to S_2$ and that F is an isometry.
- 8. (a) Let $S \subset \mathbb{R}^3$ be a regular surface and $x = x(u, v) : U \to S$ a parametrization. Define the second fundamental form II and the Gauss curvature K of S with respect to x at $p \in S$.
 - (b) Let \tilde{x} be another parametrization of S with second fundamental form II. Use the change of coordinates to express II in terms of II.
 - (c) Use (b) to prove that $\tilde{K} = K$.
 - (d) Prove that the Gauss curvature of a ruled surface is nonpositive.
- 9. (a) Let α be a regular parametrized curve in a regular surface $S \subset \mathbb{R}^3$. Define its normal curvature $\kappa_n(t)$ at the point $\alpha(t)$. Define all terms involved.
 - (b) Let x(u, v) be a regular parametrization of S, and let I, II be the associated fundamental forms, and $\alpha(t) = x(u(t), v(t))$. Prove that $\kappa_n = \frac{II(u', v')}{I(u', v')}$.
 - (c) Let $S = \{x_3 + 2x_1^2 3x_2^2 = 1\}$, $\alpha(t) = (t, -t, 1 + t^2) \subset S$. Compute the normal curvature $\kappa_n(0)$ of α at the point $\alpha(0)$ using its definition in (a).
 - (d) Let S, α be as in (c). Choose a parametrization x(u, v) of S and compute the normal curvature $\kappa_n(0)$ of α at the point $\alpha(0)$ using the identity in (b).
- 10. (a) Define what it means for a regular surface S to be compact and define the Euler characteristic of S. Define the area form of a surface S with parametrization x. State the Theorem of Gauss-Bonnet for a compact regular surface S.
 - (b) Let S be as in (a) and assume that the Euler characteristic of S vanishes. Prove that there exists a point $p \in S$ with K(p) = 0.
 - (c) State the Theorema Egregium of Gauss. Define all notions that are involved.
 - (d) Let $S_1 = \{x_1^2 4x_1x_2 + x_2^2 x_3 = 0\}$, $S_2 = \{x_1^2 + x_2^2 x_3 = 0\}$ be two regular surfaces. Use (c) to prove that there is no local isometry from S_1 to S_2 .