Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3021-WE01

Title:

Differential Geometry III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let $\alpha : (a, b) \to \mathbb{R}^2$ be a map. Define what it means for α to be a regular parametrized plane curve. Define what it means for α to be parametrized by arclength.

   (b) Prove or disprove the following claim : " $\alpha(t) = (t^2, 2t^3)$ is a regular parametrized plane curve".

   (c) Define the curvature $\kappa(t)$ of a regular parametrized plane curve $\alpha(t)$.

2. Let $\alpha(t) = (t, t^2, t^3 + 1)$ be a space curve.

   (a) Compute the curvature of α at $p = (0, 0, 1)$.

   (b) Compute the unit normal vector $N(0)$ of α at $\alpha(0)$.

3. (a) Give the definition of a regular surface $S \subset \mathbb{R}^3$.

   (b) Let $x : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$, $x(u, v) = (v \cos u, v \sin u, v)$ and let $S \subset \mathbb{R}^3$ be the image of x. Prove that S is a regular surface in $\mathbb{R}^3$.

4. Let $S \subset \mathbb{R}^3$ be a regular surface and let $x : U \to S$ be a regular parametrization of S.

   (a) Define the Gauss map on S, the Weingarten map at $p \in S$, and define what it means that $p \in S$ is elliptic/parabolic/hyperbolic.

   (b) For which $a \geq 0$ are all points of $S = \{x_1^2 + a x_2^2 = 1\} \subset \mathbb{R}^3$ parabolic ? Prove your answer.

5. Let $S = \{x_3 = x_1^2 - x_1 x_2 + 2x_2^2\ \} \subset \mathbb{R}^3$.

   (a) Show that S may be represented as $f^{-1}(b)$ for the regular value b of a smooth function f. Give a regular parametrization $x(u, v)$ of S and prove it has the required properties.

   (b) Compute the Gauss curvature of S at $p = (1, 1, 2) \in S$.

6. Consider a regular surface $S \subset \mathbb{R}^3$ parametrised by $x = x(u, v)$.

   (a) Give the definition of a geodesic on a surface.

   (b) Let S parametrized by $x(u, v) = (u^2 - v^2, v, u)$. Show that $S \cap \{x_1 = 0\}$ contains a geodesic.

SECTION B

7. (a) Let $S_1, S_2 \subset \mathbb{R}^3$ be two regular surfaces and $F : S_1 \to S_2$ be a map. Define what it means that F is differentiable at $p \in S_1$ and define $dF(p) : T_pS_1 \to T_{F(p)}S_2$.

   (b) Define what it means for F as in (a) to be an isometry.

   (c) Let $S_1 = S_2 = \{x_1^2 + x_2^2 - x_3 = 0\}$ and $F(x_1, x_2, x_3) = (-x_2, x_1, x_3)$. Prove that and $F : S_1 \to S_2$ and that F is an isometry.

8. (a) Let $S \subset \mathbb{R}^3$ be a regular surface and $x = x(u, v) : U \to S$ a parametrization. Define the second fundamental form II and the Gauss curvature K of S with respect to x at $p \in S$.

   (b) Let $\tilde{x}$ be another parametrization of S with second fundamental form $\tilde{II}$. Use the change of coordinates to express $\tilde{II}$ in terms of II.

   (c) Use (b) to prove that $\tilde{K} = K$.

   (d) Prove that the Gauss curvature of a ruled surface is nonpositive.

9. (a) Let α be a regular parametrized curve in a regular surface $S \subset \mathbb{R}^3$. Define its normal curvature $\kappa_n(t)$ at the point $\alpha(t)$. Define all terms involved.

   (b) Let $x(u, v)$ be a regular parametrization of S, and let I, II be the associated fundamental forms, and $\alpha(t) = x(u(t), v(t))$. Prove that $\kappa_n = \frac{II(u',v')}{I(u',v')}$.

   (c) Let $S = \{x_3 + 2x_1^2 - 3x_2^2 = 1\}$, $\alpha(t) = (t, -t, 1 + t^2) \subset S$. Compute the normal curvature $\kappa_n(0)$ of α at the point $\alpha(0)$ using its definition in (a).

   (d) Let S, α be as in (c). Choose a parametrization $x(u, v)$ of S and compute the normal curvature $\kappa_n(0)$ of α at the point $\alpha(0)$ using the identity in (b).

10. (a) Define what it means for a regular surface S to be compact and define the Euler characteristic of S. Define the area form of a surface S with parametrization x. State the Theorem of Gauss-Bonnet for a compact regular surface S.

    (b) Let S be as in (a) and assume that the Euler characteristic of S vanishes. Prove that there exists a point $p \in S$ with $K(p) = 0$.

    (c) State the Theorema Egregium of Gauss. Define all notions that are involved.

    (d) Let $S_1 = \{x_1^2 - 4x_1x_2 + x_2^2 - x_3 = 0\}$, $S_2 = \{x_1^2 + x_2^2 - x_3 = 0\}$ be two regular surfaces. Use (c) to prove that there is no local isometry from S_1 to S_2.