

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH3041-WE01

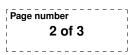
Title:

Galois Theory III

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section and the best THREE answers from S Questions in Section B carry TWICE in Section A.	ection B. as many ma	arks as those
		Dovision	1

Revision:

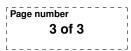


SECTION A

- 1. Find the minimal polynomials over \mathbb{Q} for:
 - (a) $\sqrt{1-\sqrt{3}};$

(b)
$$\sqrt{3} + \sqrt{7}$$
.

- 2. Find the Galois groups of the following polynomials from $\mathbb{Q}[X]$:
 - (a) $(X^4 25)^4$;
 - (b) $X^3 39X + 26$.
- 3. Let $\zeta \in \mathbb{C}$ be a primitive 25-th root of unity.
 - (a) Prove that ζ^5 is a primitive 5-th root of unity and find $Gal(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta^5))$.
 - (b) Apply Kummer's theory to prove the existence of $A \in \mathbb{Q}(\zeta^5)$ such that $\mathbb{Q}(\zeta) = \mathbb{Q}(\zeta^5)(\sqrt[5]{A})$. Find this element A.
- 4. Let K be a splitting field for $X^{11} 2$ over \mathbb{Q} .
 - (a) Prove that K contains a primitive 11-th root of unity.
 - (b) Find the degree $[K : \mathbb{Q}]$ of K over \mathbb{Q} . (Justify your answer.)
- 5. Let $P = X^5 + X^2 + 1 \in \mathbb{F}_2[X]$.
 - (a) Prove that P is irreducible.
 - (b) Let $K = \mathbb{F}_2(\theta)$, where θ is a root of P. Apply the Kummer theory to prove that $X^{31} \theta \in K[X]$ is irreducible.
- 6. Suppose L/K is a finite separable extension and E is a field between K and L.
 - (a) Prove that L/E and E/K are separable.
 - (b) Suppose K is a field of characteristic p, where p is a prime number and $F = X^p X a \in K[X]$. If θ is a root of F prove that $K(\theta)/K$ is separable.



SECTION B

- 7. Suppose L is a splitting field for $X^4 4X^2 + 11 \in \mathbb{Q}[X]$.
 - (a) Describe the structure of $G = Gal(L/\mathbb{Q})$.
 - (b) Find all subfields K in L such that [L:K] = 2 and K is Galois over \mathbb{Q} .
 - (c) Prove that there are only 3 subfields E in L such that $[E:\mathbb{Q}]=2$.
 - (d) For each field E from c), describe the structure of the group Gal(L/E).
- 8. (i) Let *L* be a splitting field for $(T^5 1)(T^{13} 1)(T^{17} 1) \in \mathbb{Q}[T]$.
 - (a) Describe the structure of $Gal(L/\mathbb{Q})$.
 - (b) Prove that there are only 7 subfields K in L such that $[K : \mathbb{Q}] = 2$. List all these subfields K.
 - (ii) Let ζ be a 13-th root of unity in \mathbb{C} .
 - (a) Find the degree $[\mathbb{Q}(\zeta \zeta^{-1}) : \mathbb{Q}].$
 - (b) Prove that $\zeta + \zeta^4 + \zeta^9 + \zeta^{16} + \zeta^{25} + \zeta^{36} = (-1 + \sqrt{13})/2.$
- 9. (i) Let *E* be a normal closure of $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{5})$.
 - (a) Describe the structure of the Galois group of E over \mathbb{Q} . (You can use without proof that $\sqrt[3]{5} \notin \mathbb{Q}(\sqrt[3]{2})$.)
 - (b) Prove that $\sqrt[3]{7} \notin \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{5})$.
 - (ii) Let $L = \mathbb{C}(X)$ and $K = \mathbb{C}(Y)$, where $Y = X^3 + (1/X^3)$ and X is a variable.
 - (a) Prove that L is Galois over K and $Gal(L/K) = S_3$ is the symmetric group of permutations of 3 symbols.
 - (b) For any subgroup H of S_3 find the corresponding subfield L^H . Let A_3 be the unique subgroup of order 3 in S_3 . Explain why there is $A \in L^{A_3}$ such that $L = L^{A_3}(\sqrt[3]{A})$. What is this element A in our case?
- 10. (i) Let $\mathbb{C}(t)$ be a field of rational functions in a variable t. Find all roots of the polynomial

$$X^{4} + (t^{2}/2)X^{2} - tX + (t^{4} + 4)/16 \in \mathbb{C}(t)[X].$$

- (ii) (a) Construct an irreducible polynomial of degree 32 in $\mathbb{F}_7[X]$.
 - (b) Let L be a splitting field for $X^4 + X^3 + 1 \in K[X]$. Compute Gal(L/K) for each of the following cases (a) $K = \mathbb{F}_2$; (b) $K = \mathbb{F}_3$; (c) $K = \mathbb{F}_4$.
 - (c) How many monic irreducible factors does $X^{511} 1 \in \mathbb{F}_2[X]$ have and what are their degrees?