Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3041-WE01

Title: Galois Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Find the minimal polynomials over $\mathbb{Q}$ for:
   (a) $\sqrt{1-\sqrt{3}}$;
   (b) $\sqrt{3}+\sqrt{7}$.

2. Find the Galois groups of the following polynomials from $\mathbb{Q}[X]$:
   (a) $(X^4-25)^4$;
   (b) $X^3-39X+26$.

3. Let $\zeta \in \mathbb{C}$ be a primitive 25-th root of unity.
   (a) Prove that ζ^5 is a primitive 5-th root of unity and find $Gal(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta^5))$.
   (b) Apply Kummer's theory to prove the existence of $A \in \mathbb{Q}(\zeta^5)$ such that $\mathbb{Q}(\zeta) = \mathbb{Q}(\zeta^5)(\sqrt[5]{A})$. Find this element A.

4. Let K be a splitting field for $X^{11}-2$ over $\mathbb{Q}$.
   (a) Prove that K contains a primitive 11-th root of unity.
   (b) Find the degree $[K:\mathbb{Q}]$ of K over $\mathbb{Q}$. (Justify your answer.)

5. Let $P = X^5+X^2+1 \in \mathbb{F}_2[X]$.
   (a) Prove that P is irreducible.
   (b) Let $K = \mathbb{F}_2(\theta)$, where θ is a root of P. Apply the Kummer theory to prove that $X^{31}-\theta \in K[X]$ is irreducible.

6. Suppose L/K is a finite separable extension and E is a field between K and L.
   (a) Prove that L/E and E/K are separable.
   (b) Suppose K is a field of characteristic p, where p is a prime number and $F = X^p - X - a \in K[X]$. If θ is a root of F prove that $K(\theta)/K$ is separable.

SECTION B

7. Suppose L is a splitting field for $X^4 - 4X^2 + 11 \in \mathbb{Q}[X]$.

   (a) Describe the structure of $G = Gal(L/\mathbb{Q})$.

   (b) Find all subfields K in L such that $[L : K] = 2$ and K is Galois over $\mathbb{Q}$.

   (c) Prove that there are only 3 subfields E in L such that $[E : \mathbb{Q}] = 2$.

   (d) For each field E from c), describe the structure of the group $Gal(L/E)$.

8. (i) Let L be a splitting field for $(T^5 - 1)(T^{13} - 1)(T^{17} - 1) \in \mathbb{Q}[T]$.

   (a) Describe the structure of $Gal(L/\mathbb{Q})$.

   (b) Prove that there are only 7 subfields K in L such that $[K : \mathbb{Q}] = 2$. List all these subfields K.

   (ii) Let ζ be a 13-th root of unity in $\mathbb{C}$.

   (a) Find the degree $[\mathbb{Q}(\zeta - \zeta^{-1}) : \mathbb{Q}]$.

   (b) Prove that $\zeta + \zeta^4 + \zeta^9 + \zeta^{16} + \zeta^{25} + \zeta^{36} = (-1 + \sqrt{13})/2$.

9. (i) Let E be a normal closure of $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{5})$.

   (a) Describe the structure of the Galois group of E over $\mathbb{Q}$. (You can use without proof that $\sqrt[3]{5} \notin \mathbb{Q}(\sqrt[3]{2})$.)

   (b) Prove that $\sqrt[3]{7} \notin \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{5})$.

   (ii) Let $L = \mathbb{C}(X)$ and $K = \mathbb{C}(Y)$, where $Y = X^3 + (1/X^3)$ and X is a variable.

   (a) Prove that L is Galois over K and $Gal(L/K) = S_3$ is the symmetric group of permutations of 3 symbols.

   (b) For any subgroup H of S_3 find the corresponding subfield L^H. Let A_3 be the unique subgroup of order 3 in S_3. Explain why there is $A \in L^{A_3}$ such that $L = L^{A_3}(\sqrt[3]{A})$. What is this element A in our case?

10. (i) Let $\mathbb{C}(t)$ be a field of rational functions in a variable t. Find all roots of the polynomial

$$X^4 + (t^2/2)X^2 - tX + (t^4 + 4)/16 \in \mathbb{C}(t)[X].$$

   (ii) (a) Construct an irreducible polynomial of degree 32 in $\mathbb{F}_7[X]$.

   (b) Let L be a splitting field for $X^4 + X^3 + 1 \in K[X]$. Compute $Gal(L/K)$ for each of the following cases (a) $K = \mathbb{F}_2$; (b) $K = \mathbb{F}_3$; (c) $K = \mathbb{F}_4$.

   (c) How many monic irreducible factors does $X^{511} - 1 \in \mathbb{F}_2[X]$ have and what are their degrees?