Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3051-WE01

Title:

Statistical Methods III

Time Allowed:	2 hours 30 minutes	
Additional Material provided:	Tables: Normal, t-distribution, F-distribution, χ^2-distribution; Graph paper.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **THREE** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Consider the linear regression model $y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i, i = 1, \ldots, n$, with $\boldsymbol{\beta} \in \mathbb{R}^p$, and independent errors $\epsilon_i \sim N(0, \sigma^2)$. Denote by $\hat{\boldsymbol{\beta}}$ the least squares estimate of $\boldsymbol{\beta}$. We are now given a new vector $\boldsymbol{x}_0 \in \mathbb{R}^p$ of predictor values, and we are interested in the uncertainty associated with the prediction

$$\hat{y}_0 = \boldsymbol{x}_0^T \hat{\boldsymbol{\beta}}$$

of the true, unknown response value y_0. You can use without proof that $\mathrm{Var}(\hat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}$, with $\boldsymbol{X}$ denoting the design matrix.

(a) Derive an expression for the variance of prediction, that is $\mathrm{Var}(y_0 - \hat{y}_0)$. Hence, by replacing σ^2 by its unbiased estimate s^2, and taking a square root of the resulting expression, give an expression for the *prediction error*.

(b) Write down the expression for a $(1-\alpha)$ prediction interval for y_0.

(c) Explain qualitatively why, in the case $n \longrightarrow \infty$, the prediction interval from part (b) can be approximated by

$$\boldsymbol{x}_0^T \hat{\boldsymbol{\beta}} \pm z_{\alpha/2} s \qquad (1)$$

where $z_{\alpha/2}$ is the Gaussian quantile with right tail probability $\alpha/2$.

(d) Is the interval defined by (1) generally smaller, or generally wider, than its original counterpart from part (b)? Explain your answer carefully, and also explain your reasoning if no general statement can be made.

2. (a) Let $\mathcal{A}$ be a factor with a levels $\{1,\ldots,a\}$. We consider a regression problem of type

$$E[y|\mathcal{A}] = \beta_0 + \beta_1 x_1^{\mathcal{A}} + \cdots + \beta_{a-1} x_{a-1}^{\mathcal{A}}, \tag{2}$$

using dummy coding $x_j^{\mathcal{A}} = 1_{\{\mathcal{A}=j\}}$, and with the last category a serving as reference category. Let $\mu_j = E(y|\mathcal{A}=j)$.

Write the expectations μ_j as functions of the β_j. From this, derive expressions for the parameters β_j, $j=0,\ldots,a-1$ as functions of the expectations μ_j, $j=1,\ldots,a$. Interpret the result briefly.

(b) An experiment was conducted to assess the potency of various constituents of orchard sprays in repelling honeybees. Individual cells of dry comb were filled with a measured amount of lime sulphur emulsion in sucrose solution. Seven different concentrations of lime sulphur (labelled A,B, ..., G) ranging from a concentration of 1/100 to 1/1,562,500 were used, in addition to a solution containing no sulphur (labelled H) [In the notation from part (a), it is clear that factor levels $1,\ldots,a$ just correspond to concentration types $A,\ldots,H$].

The responses for the eight different solutions were obtained by releasing 100 bees into the chamber for two hours, and then measuring the decrease in volume of the solutions in the individual cells. Eight replicates were obtained per concentration type, resulting in the following data set:

Concentration	Reduction of volume							
A	2	2	5	4	5	12	4	3
B	8	6	4	10	7	4	8	14
C	15	84	16	9	17	29	13	19
D	57	36	22	51	28	27	20	39
E	95	51	39	114	43	47	61	55
F	90	69	87	20	71	44	57	114
G	92	71	72	24	60	77	72	80
H	69	127	72	130	81	76	81	86

For this data set, a linear model of type (2) is fitted. Give the estimates $\hat\beta_0, \hat\beta_1$ and $\hat\beta_7$. Explain your working.

3. Consider a standard linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with p predictors and an intercept, with $E[\boldsymbol{\epsilon}] = \mathbf{0}$ and $\text{Var}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix. In what follows you may use the result

$$\text{Cov}(\boldsymbol{AY}, \boldsymbol{BY}) = \boldsymbol{A}\text{Var}(\boldsymbol{Y})\boldsymbol{B}^T$$

where $\boldsymbol{Y}$ is any random vector and $\boldsymbol{A}$ and $\boldsymbol{B}$ are known matrices.

(a) Given that the least squares estimator of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$,

(i) show that the vector of **residuals** $\hat{\boldsymbol{\epsilon}} = \boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}$ and the vector of **fitted values** $\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}$ can be written as $\hat{\boldsymbol{\epsilon}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$ and $\hat{\boldsymbol{Y}} = \boldsymbol{HY}$, where $\boldsymbol{H} = (h_{ij})$ is the **hat matrix** $\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$;

(ii) show that $\text{Var}(\hat{\epsilon}_i) = (1 - h_{ii})\sigma^2$ and $\text{Cov}(\hat{\epsilon}_i, \hat{y}_i) = 0$, and discuss briefly practical uses of these results.

You may assume that $\boldsymbol{H}^2 = \boldsymbol{H} = \boldsymbol{H}^T$.

(b) Discuss briefly the terms "influential", "potentially influential" (leverage), and "outlier" in the context of a standard linear regression model with p predictors including an intercept, distinguishing carefully between model diagnosis and robustness of conclusions. In your discussion refer to the terms in the following expression for Cook's distance D_i (for case $i = 1, 2, ..., n$)

$$D_i = \frac{r_i^2}{p} \frac{h_{ii}}{1 - h_{ii}}$$

where h_{ii} is leverage and r_i is the "internally studentised residual".

4. A principal component analysis of a 4-dimensional data set (containing the annual power generations of four power stations over a 20 years period) has been carried out. The standard deviation of each principal component (PC) is listed below:

	PC1	PC2	PC3	PC4
Standard deviation	2.056	0.492	0.279	0.154

The rotation matrix resulting from the principal component analysis is as follows:

	PC1	PC2	PC3	PC4
Station 1	0.361	-0.656	0.582	0.315
Station 2	-0.084	-0.730	-0.598	-0.319
Station 3	0.856	0.173	-0.076	-0.479
Station 4	0.358	0.075	-0.545	0.753

Let $\hat{\boldsymbol{\Sigma}} \in \mathbb{R}^{4\times 4}$ denote the sample variance matrix of the data set.

(a) Outline the main goals of principal components analysis of data.

(b) Give the ordered eigenvalues of $\hat{\boldsymbol{\Sigma}}$, and find the total variance of the data cloud.

(c) Draw a scree plot. How many components would you need in order to capture at least 95% of the total variance of the data cloud?

(d) Give the first and the second eigenvector of $\hat{\boldsymbol{\Sigma}}$, which we denote by $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. Without carrying out calculations, give the numerical values of $\boldsymbol{\gamma}_1^T\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2^T\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2^T\boldsymbol{\gamma}_2$, and explain your answer.

SECTION B

5. We are given monthly records $\boldsymbol{x}_i = (x_{i1}, x_{i2})^T, i = 1, \ldots, 12$, of excess returns of the durables industry (X_1) and construction industry (X_2) in the year 2002:

i	1	2	3	4	5	6
x_{i1}	-3.82	3.09	3.58	-0.91	-1.26	-7.85
x_{i2}	-0.20	1.52	-0.47	-0.85	-3.46	-5.77
i	7	8	9	10	11	12
x_{i1}	-12.97	1.87	-11.07	4.58	15.63	-4.89
x_{i2}	-12.69	1.85	-12.58	5.00	0.99	-5.13

You can use that $\bar{x}_1 = -1.168$, $\bar{x}_2 = -2.649$, $\sum x_{i1}^2 = 684.4$, $\sum x_{i2}^2 = 423.6$, $\sum x_{i1}x_{i2} = 425.0$.

(a) Compute the sample variance matrix $\hat{\boldsymbol{\Sigma}}$.

(b) Assuming bivariate normality of $X = (X_1, X_2)^T$, explain how the maximum likelihood estimator $\hat{\boldsymbol{\Sigma}}_{ML}$ of $\boldsymbol{\Sigma} = \text{Var}(X)$ is related to your result from (a) [without setting up any likelihood function or solving equations, etc.], and compute this matrix for the given data explicitly.

Suppose now that we make a more specific assumption about the distribution of X, namely

$$X \sim N_2(\mathbf{0}, \boldsymbol{\Sigma}), \qquad \text{with} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2 & r\sigma^2 \\ r\sigma^2 & \sigma^2 \end{pmatrix}.$$

(c) Write down the probability density function of X, simplifying the expression as far as possible.

(d) Assuming the data $\boldsymbol{x}_i$, $i = 1, \ldots, n$ to be a random sample from this distribution, find the log-likelihood function $L(r, \sigma)$.

(e) Show that the maximum likelihood estimates of r and σ are given by

$$\hat{r} = 2\frac{\sum x_{i1}x_{i2}}{\sum x_{i1}^2 + \sum x_{i2}^2}$$

and

$$\hat{\sigma}^2 = \frac{1}{2n(1-\hat{r}^2)} \sum (x_{i1}^2 - 2\hat{r}x_{i1}x_{i2} + x_{i2}^2).$$

(f) Using the result from (e), compute now the variance matrix for the given data. Compare with your result from (b), commenting on the similarity / discrepancy between the two results.

6. Consider a linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{Y} = (y_1, \ldots, y_n)^T$, and $\boldsymbol{\beta} \in \mathbb{R}^p$.

   (a) Assume that $\boldsymbol{Y}$ fulfils the standard linear model assumptions, that is we have normally distributed and homoscedastic errors $\boldsymbol{\epsilon} \sim N_n(\boldsymbol{0}, \sigma^2\boldsymbol{I})$, with $\boldsymbol{I}$ being the $n \times n$ identity matrix. Denote, as usual,

   $$s^2 = \frac{(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})^T(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-p},$$

   where $\hat{\boldsymbol{\beta}}$ is the least squares regression estimate of $\boldsymbol{\beta}$.

      (i) Provide a heuristic argument which demonstrates that the sampling distribution of $(n-p)s^2/\sigma^2$ is given by a χ^2 distribution with $n-p$ degrees of freedom.
      [*Note:* You can make use of the decomposition

      $$(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{\beta}) = (\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})^T(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}) + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}).$$

      You do not need to produce any proofs, but state all steps, equations, and properties that are required for your reasoning.]

      (ii) From the result of (i), or otherwise, show that s^2 is an unbiased estimator for σ^2.

      (iii) Show that

      $$E(s) \;\leq\; \sigma,$$

      demonstrating that s is in general a negatively biased estimate of σ.

   (b) Considering the case of *positive* responses $y_i > 0$, suppose we seek some power transformation of the form

   $$y_i^{(\lambda)} = \begin{cases} \frac{y_i^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \log y_i & \lambda = 0 \end{cases}$$

   such that the transformed response vector $\boldsymbol{Y}^{(\lambda)} = (y_1^{(\lambda)}, ..., y_n^{(\lambda)})^T$ satisfies the standard linear model assumptions.

      (i) Assuming $y_i^{(\lambda)} \sim N(\boldsymbol{x}_i^T\boldsymbol{\beta}, \sigma^2)$, write down the probability density function $f(y_i^{(\lambda)}|\boldsymbol{x}_i)$. From this, find the probability density function $f(y_i|\boldsymbol{x}_i)$.

      (ii) Consequently, show that the log-likelihood for $\boldsymbol{\beta}, \sigma, \lambda$ based on independent $y_1, ..., y_n$ is given by

      $$L(\boldsymbol{\beta}, \sigma, \lambda) = -n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i^{(\lambda)} - \boldsymbol{x}_i^T\boldsymbol{\beta})^2 + (\lambda - 1)\sum_{i=1}^n \log y_i,$$

      plus an additive constant not depending on $\boldsymbol{\beta}, \sigma, \lambda$.

      (iii) If λ is regarded as fixed, it is clear that the maximum likelihood estimate $\hat{\boldsymbol{\beta}}_\lambda$ of $\boldsymbol{\beta}$ takes the usual form but with the response vector $\boldsymbol{Y}$ replaced by $\boldsymbol{Y}^{(\lambda)}$, and that the maximum likelihood estimator of σ^2 is given by

      $$\hat{\sigma}_\lambda^2 = \frac{1}{n}\sum_{i=1}^n \left(y_i^{(\lambda)} - \boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}_\lambda\right)^2$$

      (you don't need to show this). Find an expression for the profile log likelihood $L_p(\lambda) = L(\hat{\boldsymbol{\beta}}_\lambda, \hat{\sigma}_\lambda, \lambda)$, and simplify as far as possible.

7. There is considerable interest in modelling vehicle fuel consumption. For this study, the fuel consumption FUEL (gallons per person) was recorded in 48 US states. The covariates are TAX (cents per gallon), DLIC (% population with driving licenses), INC (average income in $\$1000's$), and ROAD (1000's of miles of highway). The first two columns of a sequential Analysis of Variance table are provided below.

	df	SS
TAX	1	119823
DLIC	1	207709
INC	1	69532
ROAD	1	2252
Error	43	189050

(a) At the 1% level of significance, carry out appropriate F-tests which address the following questions:

(i) Do the four covariates, as a whole, contribute significantly to the variation of fuel consumption?

(ii) Does TAX contribute significantly to the variation of fuel consumption?

(iii) Given the inclusion of TAX, DLIC, and INC, does ROAD contribute significantly to the variation of fuel consumption?

(b) A well-known model selection criterion is Mallows' $C_{\mathcal{I}}$, which is given by $C_{\mathcal{I}} = \frac{RSS_{\mathcal{I}}}{s^2} + 2p_{\mathcal{I}} - n$, where $\mathcal{I}$ refers to the index set of variables included in the model, and $p_{\mathcal{I}}$ denotes the cardinality of $\mathcal{I}$.

(i) Show that Mallows' $C_{\mathcal{I}}$ can be expressed in the form $a_{\mathcal{I}} + b_{\mathcal{I}} F_{\mathcal{D}}$, where $a_{\mathcal{I}}$ and $b_{\mathcal{I}}$ are constants which depend on $\mathcal{I}$ through $p_{\mathcal{I}}$, $\mathcal{D}$ is the index set of variables *not* included in the model, and $F_{\mathcal{D}}$ is the test statistic for testing H_0: 'Given the inclusion of $\mathcal{I}$, the variables in $\mathcal{D}$ do not contribute to the variation in the response'. Use the connection to the F-test to deduce which values of $C_{\mathcal{I}}$ one can expect for 'good' submodels.

(ii) For the data set considered, compute $C_{\mathcal{I}}$ for the submodels:

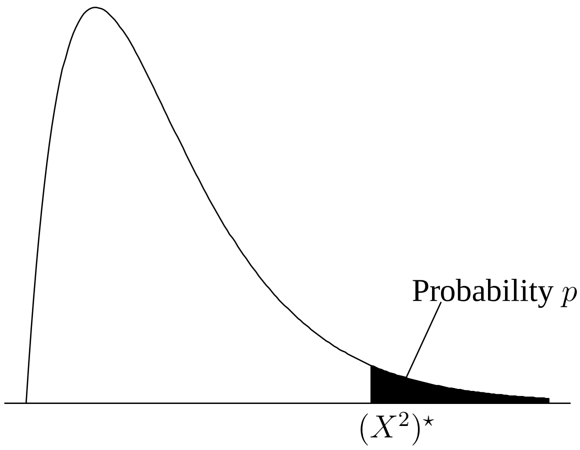
- A: 1+TAX+DLIC,
- B: 1+TAX+DLIC+INC
- C: 1+TAX+DLIC+INC+ROAD.

Which (if any) of the three models would you select based on this analysis?

(iii) If you had to carry out a full model selection procedure, how many submodels would there be to consider if you always include an intercept? Name two strategies for reducing the number of submodels to consider, and explain briefly what they do.

Probabilities for the χ^2-distribution

Table entry for p is the point $(X^2)^\star$ with probability p lying above it



	Tail probability p											
df	.995	.975	.25	.2	.1	.05	.025	.01	.005	.0025	.001	.0005
1	0.000039	0.00098	1.32	1.64	2.71	3.84	5.02	6.63	7.88	9.14	10.83	12.12
2	0.010	0.051	2.77	3.22	4.61	5.99	7.38	9.21	10.60	11.98	13.82	15.20
3	0.072	0.22	4.11	4.64	6.25	7.81	9.35	11.34	12.84	14.32	16.27	17.73
4	0.21	0.48	5.39	5.99	7.78	9.49	11.14	13.28	14.86	16.42	18.47	20.00
5	0.41	0.83	6.63	7.29	9.24	11.07	12.83	15.09	16.75	18.39	20.52	22.11
6	0.68	1.24	7.84	8.56	10.64	12.59	14.45	16.81	18.55	20.25	22.46	24.10
7	0.99	1.69	9.04	9.80	12.02	14.07	16.01	18.48	20.28	22.04	24.32	26.02
8	1.34	2.18	10.22	11.03	13.36	15.51	17.53	20.09	21.95	23.77	26.12	27.87
9	1.73	2.70	11.39	12.24	14.68	16.92	19.02	21.67	23.59	25.46	27.88	29.67
10	2.16	3.25	12.55	13.44	15.99	18.31	20.48	23.21	25.19	27.11	29.59	31.42
11	2.60	3.82	13.70	14.63	17.28	19.68	21.92	24.72	26.76	28.73	31.26	33.14
12	3.07	4.40	14.85	15.81	18.55	21.03	23.34	26.22	28.30	30.32	32.91	34.82
13	3.57	5.01	15.98	16.98	19.81	22.36	24.74	27.69	29.82	31.88	34.53	36.48
14	4.07	5.63	17.12	18.15	21.06	23.68	26.12	29.14	31.32	33.43	36.12	38.11
15	4.60	6.26	18.25	19.31	22.31	25.00	27.49	30.58	32.80	34.95	37.70	39.72
16	5.14	6.91	19.37	20.47	23.54	26.30	28.85	32.00	34.27	36.46	39.25	41.31
17	5.70	7.56	20.49	21.61	24.77	27.59	30.19	33.41	35.72	37.95	40.79	42.88
18	6.26	8.23	21.60	22.76	25.99	28.87	31.53	34.81	37.16	39.42	42.31	44.43
19	6.84	8.91	22.72	23.90	27.20	30.14	32.85	36.19	38.58	40.88	43.82	45.97
20	7.43	9.59	23.83	25.04	28.41	31.41	34.17	37.57	40.00	42.34	45.31	47.50
21	8.03	10.28	24.93	26.17	29.62	32.67	35.48	38.93	41.40	43.78	46.80	49.01
22	8.64	10.98	26.04	27.30	30.81	33.92	36.78	40.29	42.80	45.20	48.27	50.51
23	9.26	11.69	27.14	28.43	32.01	35.17	38.08	41.64	44.18	46.62	49.73	52.00
24	9.89	12.40	28.24	29.55	33.20	36.42	39.36	42.98	45.56	48.03	51.18	53.48
25	10.52	13.12	29.34	30.68	34.38	37.65	40.65	44.31	46.93	49.44	52.62	54.95
26	11.16	13.84	30.43	31.79	35.56	38.89	41.92	45.64	48.29	50.83	54.05	56.41
27	11.81	14.57	31.53	32.91	36.74	40.11	43.19	46.96	49.64	52.22	55.48	57.86
28	12.46	15.31	32.62	34.03	37.92	41.34	44.46	48.28	50.99	53.59	56.89	59.30
29	13.12	16.05	33.71	35.14	39.09	42.56	45.72	49.59	52.34	54.97	58.30	60.73
30	13.79	16.79	34.80	36.25	40.26	43.77	46.98	50.89	53.67	56.33	59.70	62.16
40	20.71	24.43	45.62	47.27	51.81	55.76	59.34	63.69	66.77	69.70	73.40	76.09
50	27.99	32.36	56.33	58.16	63.17	67.50	71.42	76.15	79.49	82.66	86.66	89.56
60	35.53	40.48	66.98	68.97	74.40	79.08	83.30	88.38	91.95	95.34	99.61	102.69
80	51.17	57.15	88.13	90.41	96.58	101.88	106.63	112.33	116.32	120.10	124.84	128.26
100	67.33	74.22	109.14	111.67	118.50	124.34	129.56	135.81	140.17	144.29	149.45	153.17

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
2	0.1	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	0.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	0.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	0.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	0.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	0.1	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	0.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	0.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	0.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	0.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	0.1	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	0.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	0.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	0.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	0.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	0.1	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	0.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	0.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	0.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	0.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24
6	0.1	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	0.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	0.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	0.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	0.001	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69
7	0.1	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	0.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	0.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	0.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	0.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33
8	0.1	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
	0.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
	0.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
	0.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
	0.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77
9	0.1	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
	0.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
	0.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
	0.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
	0.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11
10	0.1	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
	0.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
	0.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
	0.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
	0.001	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96
11	0.1	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27
	0.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
	0.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59
	0.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
	0.001	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12
12	0.1	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
	0.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
	0.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
	0.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
	0.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48
13	0.1	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	0.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	0.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
	0.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	0.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
14	0.1	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	0.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	0.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
	0.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	0.1	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	0.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	0.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
	0.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	0.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	0.1	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	0.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	0.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
	0.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	0.1	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	0.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	0.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
	0.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
	0.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75
18	0.1	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	0.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
	0.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
	0.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56
19	0.1	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	0.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
	0.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
	0.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	0.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39
20	0.1	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	0.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
	0.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
	0.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24
21	0.1	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
	0.05	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
	0.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80
	0.01	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
	0.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11
22	0.1	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	0.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	0.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
	0.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	0.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99
23	0.1	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	0.05	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
	0.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73
	0.01	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	0.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89
24	0.1	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	0.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
	0.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
	0.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	0.001	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80
25	0.1	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
	0.05	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
	0.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68
	0.01	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
	0.001	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71

F distribution critical values

Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
26	0.1	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
	0.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
	0.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
	0.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
	0.001	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64
27	0.1	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
	0.05	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
	0.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63
	0.01	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
	0.001	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57
28	0.1	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
	0.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
	0.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
	0.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
	0.001	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50
29	0.1	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
	0.05	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
	0.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59
	0.01	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
	0.001	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45
30	0.1	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
	0.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
	0.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
	0.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
	0.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39
40	0.1	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
	0.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
	0.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
	0.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
	0.001	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02
50	0.1	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76
	0.05	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
	0.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
	0.01	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78
	0.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82
60	0.1	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
	0.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
	0.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
	0.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
	0.001	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69
100	0.1	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69
	0.05	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
	0.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
	0.01	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
	0.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44
200	0.1	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66
	0.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
	0.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
	0.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
	0.001	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26
1000	0.1	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.64
	0.05	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
	0.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13
	0.01	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43
	0.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13

F distribution critical values

Denominator degrees of freedom	p	10	11	12	13	14	15	16	17	18	19
		Degrees of freedom in the numerator									
	0.1	9.39	9.40	9.41	9.41	9.42	9.42	9.43	9.43	9.44	9.44
	0.05	19.40	19.40	19.41	19.42	19.42	19.43	19.43	19.44	19.44	19.44
2	0.025	39.40	39.41	39.41	39.42	39.43	39.43	39.44	39.44	39.44	39.45
	0.01	99.40	99.41	99.42	99.42	99.43	99.43	99.44	99.44	99.44	99.45
	0.001	999.40	999.41	999.42	999.42	999.43	999.43	999.44	999.44	999.44	999.45
	0.1	5.23	5.22	5.22	5.21	5.20	5.20	5.20	5.19	5.19	5.19
	0.05	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67
3	0.025	14.42	14.37	14.34	14.30	14.28	14.25	14.23	14.21	14.20	14.18
	0.01	27.23	27.13	27.05	26.98	26.92	26.87	26.83	26.79	26.75	26.72
	0.001	129.25	128.74	128.32	127.96	127.64	127.37	127.14	126.93	126.74	126.57
	0.1	3.92	3.91	3.90	3.89	3.88	3.87	3.86	3.86	3.85	3.85
	0.05	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81
4	0.025	8.84	8.79	8.75	8.71	8.68	8.66	8.63	8.61	8.59	8.58
	0.01	14.55	14.45	14.37	14.31	14.25	14.20	14.15	14.11	14.08	14.05
	0.001	48.05	47.70	47.41	47.16	46.95	46.76	46.60	46.45	46.32	46.21
	0.1	3.30	3.28	3.27	3.26	3.25	3.24	3.23	3.22	3.22	3.21
	0.05	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57
5	0.025	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34
	0.01	10.05	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61	9.58
	0.001	26.92	26.65	26.42	26.22	26.06	25.91	25.78	25.67	25.57	25.48
	0.1	2.94	2.92	2.90	2.89	2.88	2.87	2.86	2.85	2.85	2.84
	0.05	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88
6	0.025	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18
	0.01	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45	7.42
	0.001	18.41	18.18	17.99	17.82	17.68	17.56	17.45	17.35	17.27	17.19
	0.1	2.70	2.68	2.67	2.65	2.64	2.63	2.62	2.61	2.61	2.60
	0.05	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46
7	0.025	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48
	0.01	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18
	0.001	14.08	13.88	13.71	13.56	13.43	13.32	13.23	13.14	13.06	12.99
	0.1	2.54	2.52	2.50	2.49	2.48	2.46	2.45	2.45	2.44	2.43
	0.05	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16
8	0.025	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02
	0.01	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38
	0.001	11.54	11.35	11.19	11.06	10.94	10.84	10.75	10.67	10.60	10.54
	0.1	2.42	2.40	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.30
	0.05	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95
9	0.025	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68
	0.01	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83
	0.001	9.89	9.72	9.57	9.44	9.33	9.24	9.15	9.08	9.01	8.95
	0.1	2.32	2.30	2.28	2.27	2.26	2.24	2.23	2.22	2.22	2.21
	0.05	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79
10	0.025	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44
	0.01	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43
	0.001	8.75	8.59	8.45	8.32	8.22	8.13	8.05	7.98	7.91	7.86
	0.1	2.25	2.23	2.21	2.19	2.18	2.17	2.16	2.15	2.14	2.13
	0.05	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66
11	0.025	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24
	0.01	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12
	0.001	7.92	7.76	7.63	7.51	7.41	7.32	7.24	7.17	7.11	7.06
	0.1	2.19	2.17	2.15	2.13	2.12	2.10	2.09	2.08	2.08	2.07
	0.05	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56
12	0.025	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09
	0.01	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88
	0.001	7.29	7.14	7.00	6.89	6.79	6.71	6.63	6.57	6.51	6.45
	0.1	2.14	2.12	2.10	2.08	2.07	2.05	2.04	2.03	2.02	2.01
	0.05	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47
13	0.025	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96
	0.01	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69
	0.001	6.80	6.65	6.52	6.41	6.31	6.23	6.16	6.09	6.03	5.98

F distribution critical values

Denominator degrees of freedom		Degrees of freedom in the numerator									
	p	10	11	12	13	14	15	16	17	18	19
14	0.1	2.10	2.07	2.05	2.04	2.02	2.01	2.00	1.99	1.98	1.97
	0.05	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40
	0.025	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86
	0.01	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53
	0.001	6.40	6.26	6.13	6.02	5.93	5.85	5.78	5.71	5.66	5.60
15	0.1	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93
	0.05	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34
	0.025	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77
	0.01	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40
	0.001	6.08	5.94	5.81	5.71	5.62	5.54	5.46	5.40	5.35	5.29
16	0.1	2.03	2.01	1.99	1.97	1.95	1.94	1.93	1.92	1.91	1.90
	0.05	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29
	0.025	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70
	0.01	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28
	0.001	5.81	5.67	5.55	5.44	5.35	5.27	5.20	5.14	5.09	5.04
17	0.1	2.00	1.98	1.96	1.94	1.93	1.91	1.90	1.89	1.88	1.87
	0.05	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24
	0.025	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63
	0.01	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19
	0.001	5.58	5.44	5.32	5.22	5.13	5.05	4.99	4.92	4.87	4.82
18	0.1	1.98	1.95	1.93	1.92	1.90	1.89	1.87	1.86	1.85	1.84
	0.05	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20
	0.025	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58
	0.01	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10
	0.001	5.39	5.25	5.13	5.03	4.94	4.87	4.80	4.74	4.68	4.63
19	0.1	1.96	1.93	1.91	1.89	1.88	1.86	1.85	1.84	1.83	1.82
	0.05	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17
	0.025	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53
	0.01	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03
	0.001	5.22	5.08	4.97	4.87	4.78	4.70	4.64	4.58	4.52	4.47
20	0.1	1.94	1.91	1.89	1.87	1.86	1.84	1.83	1.82	1.81	1.80
	0.05	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14
	0.025	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48
	0.01	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96
	0.001	5.08	4.94	4.82	4.72	4.64	4.56	4.49	4.44	4.38	4.33
21	0.1	1.92	1.90	1.87	1.86	1.84	1.83	1.81	1.80	1.79	1.78
	0.05	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11
	0.025	2.73	2.68	2.64	2.60	2.56	2.53	2.51	2.48	2.46	2.44
	0.01	3.31	3.24	3.17	3.12	3.07	3.03	2.99	2.96	2.93	2.90
	0.001	4.95	4.81	4.70	4.60	4.51	4.44	4.37	4.31	4.26	4.21
22	0.1	1.90	1.88	1.86	1.84	1.83	1.81	1.80	1.79	1.78	1.77
	0.05	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08
	0.025	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41
	0.01	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85
	0.001	4.83	4.70	4.58	4.49	4.40	4.33	4.26	4.20	4.15	4.10
23	0.1	1.89	1.87	1.84	1.83	1.81	1.80	1.78	1.77	1.76	1.75
	0.05	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06
	0.025	2.67	2.62	2.57	2.53	2.50	2.47	2.44	2.42	2.39	2.37
	0.01	3.21	3.14	3.07	3.02	2.97	2.93	2.89	2.86	2.83	2.80
	0.001	4.73	4.60	4.48	4.39	4.30	4.23	4.16	4.10	4.05	4.00
24	0.1	1.88	1.85	1.83	1.81	1.80	1.78	1.77	1.76	1.75	1.74
	0.05	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04
	0.025	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35
	0.01	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76
	0.001	4.64	4.51	4.39	4.30	4.21	4.14	4.07	4.02	3.96	3.92
25	0.1	1.87	1.84	1.82	1.80	1.79	1.77	1.76	1.75	1.74	1.73
	0.05	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02
	0.025	2.61	2.56	2.51	2.48	2.44	2.41	2.38	2.36	2.34	2.32
	0.01	3.13	3.06	2.99	2.94	2.89	2.85	2.81	2.78	2.75	2.72
	0.001	4.56	4.42	4.31	4.22	4.13	4.06	3.99	3.94	3.88	3.84

F distribution critical values

Denominator degrees of freedom	p	Degrees of freedom in the numerator: 10	11	12	13	14	15	16	17	18	19
26	0.1	1.86	1.83	1.81	1.79	1.77	1.76	1.75	1.73	1.72	1.71
	0.05	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00
	0.025	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29
	0.01	3.09	3.02	2.96	2.90	2.86	2.81	2.78	2.75	2.72	2.69
	0.001	4.48	4.35	4.24	4.14	4.06	3.99	3.92	3.86	3.81	3.77
27	0.1	1.85	1.82	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70
	0.05	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99
	0.025	2.57	2.51	2.47	2.43	2.39	2.36	2.34	2.31	2.29	2.27
	0.01	3.06	2.99	2.93	2.87	2.82	2.78	2.75	2.71	2.68	2.66
	0.001	4.41	4.28	4.17	4.08	3.99	3.92	3.86	3.80	3.75	3.70
28	0.1	1.84	1.81	1.79	1.77	1.75	1.74	1.73	1.71	1.70	1.69
	0.05	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97
	0.025	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25
	0.01	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63
	0.001	4.35	4.22	4.11	4.01	3.93	3.86	3.80	3.74	3.69	3.64
29	0.1	1.83	1.80	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.68
	0.05	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96
	0.025	2.53	2.48	2.43	2.39	2.36	2.32	2.30	2.27	2.25	2.23
	0.01	3.00	2.93	2.87	2.81	2.77	2.73	2.69	2.66	2.63	2.60
	0.001	4.29	4.16	4.05	3.96	3.88	3.80	3.74	3.68	3.63	3.59
30	0.1	1.82	1.79	1.77	1.75	1.74	1.72	1.71	1.70	1.69	1.68
	0.05	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95
	0.025	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21
	0.01	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57
	0.001	4.24	4.11	4.00	3.91	3.82	3.75	3.69	3.63	3.58	3.53
40	0.1	1.76	1.74	1.71	1.70	1.68	1.66	1.65	1.64	1.62	1.61
	0.05	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87	1.85
	0.025	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09
	0.01	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42	2.39
	0.001	3.87	3.75	3.64	3.55	3.47	3.40	3.34	3.28	3.23	3.19
50	0.1	1.73	1.70	1.68	1.66	1.64	1.63	1.61	1.60	1.59	1.58
	0.05	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81	1.80
	0.025	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03	2.01
	0.01	2.70	2.63	2.56	2.51	2.46	2.42	2.38	2.35	2.32	2.29
	0.001	3.67	3.55	3.44	3.35	3.27	3.20	3.14	3.09	3.04	2.99
60	0.1	1.71	1.68	1.66	1.64	1.62	1.60	1.59	1.58	1.56	1.55
	0.05	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78	1.76
	0.025	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96
	0.01	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25	2.22
	0.001	3.54	3.42	3.32	3.23	3.15	3.08	3.02	2.96	2.91	2.87
100	0.1	1.66	1.64	1.61	1.59	1.57	1.56	1.54	1.53	1.52	1.50
	0.05	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73	1.71	1.69
	0.025	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89	1.87
	0.01	2.50	2.43	2.37	2.31	2.27	2.22	2.19	2.15	2.12	2.09
	0.001	3.30	3.18	3.07	2.99	2.91	2.84	2.78	2.73	2.68	2.63
200	0.1	1.63	1.60	1.58	1.56	1.54	1.52	1.51	1.49	1.48	1.47
	0.05	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67	1.66	1.64
	0.025	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82	1.80
	0.01	2.41	2.34	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.00
	0.001	3.12	3.00	2.90	2.82	2.74	2.67	2.61	2.56	2.51	2.46
1000	0.1	1.61	1.58	1.55	1.53	1.51	1.49	1.48	1.46	1.45	1.44
	0.05	1.84	1.80	1.76	1.73	1.70	1.68	1.65	1.63	1.61	1.60
	0.025	2.06	2.01	1.96	1.92	1.88	1.85	1.82	1.79	1.77	1.74
	0.01	2.34	2.27	2.20	2.15	2.10	2.06	2.02	1.98	1.95	1.92
	0.001	2.99	2.87	2.77	2.69	2.61	2.54	2.48	2.43	2.38	2.34

Probabilities for the standard normal distribution

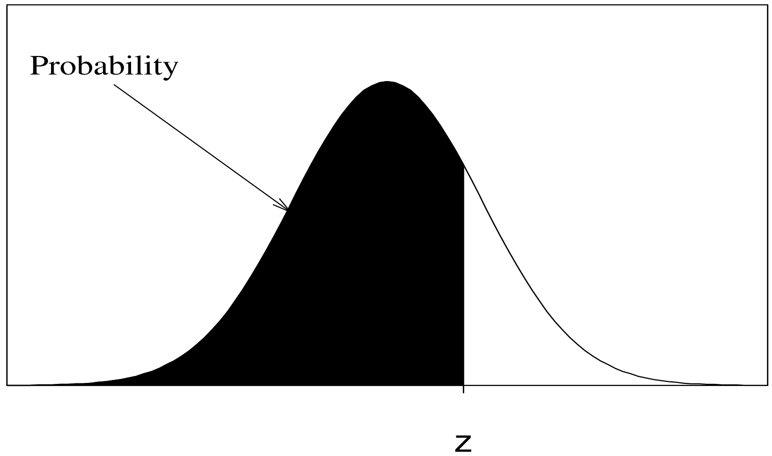
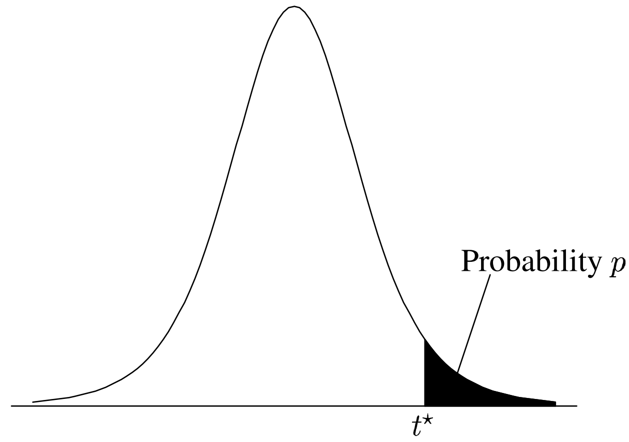


Table entry for z is the probability lying to the left of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Probabilities for the t-distribution

Table entry for p and C is the point $t^\star$ with probability p lying above it and probability C lying between $-t^\star$ and $t^\star$



	Tail probability p											
df	.25	.2	.15	.1	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.895	31.821	63.657	127.321	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70 %	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											