Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3071-WE01

Title:

Decision Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let $\mathcal{G}$ be the set of gambles over a set of basic rewards $\mathcal{R}$, and $\geq^*$ a preference order on $\mathcal{G}$. State the conditions that must be satisfied by a function $U : \mathcal{G} \to \mathbb{R}$ for it to be a utility function for $\mathcal{G}$ and $\geq^*$.

   (b) Explain the relevance of these conditions for decision making under uncertainty.

   (c) Describe (without proof) a procedure to construct a utility function for the set of gambles and a preference order over a set of rewards.

   (d) Give one condition that the preference order must satisfy for this procedure to work.

   (e) Explain to what extent a utility function representing a given set of preferences on a set of rewards is unique.

2. (a) Individuals A and B have utilities for non-negative amounts of money of the form $U_{\mathrm{A}}(£x) = \log(x + a)$ and $U_{\mathrm{B}}(£x) = \log(x + b)$, where $a < b$. Discuss and compare the attitudes to risk of A and B. (Any results about risk attitudes that you require should be stated clearly, but need not be proved.)

   (b) Suppose that individuals A and B each have a utility for money of the form $U(£x) = \log(x + 1)$. Individual A currently has no money and B has £7. Individual A has a raffle ticket that, with probability 1/2 will pay £8 and with probability 1/2 will pay nothing. Show that there is an amount $£t$ that B would be prepared to pay for the ticket and which A would be prepared to accept.

3. Define what it means for attributes X and Y to be mutually utility independent with respect to a set of preferences or a utility function.

   Suppose that you may receive two amounts of money. You receive M_1 immediately and you receive M_2 in three years' time. Suppose that you consider M_1 and M_2 to be mutually utility independent. Your marginal utilities for M_1 and M_2 are both of the form $U(£m) = \sqrt{m}$ for non-negative m.

   Suppose that you are indifferent between the three choices:

   (i) $(m_1, m_2) = (16, 0)$;

   (ii) $(m_1, m_2) = (0, 64)$;

   (iii) $(m_1, m_2) = (9, 36)$.

   With origin $(m_1, m_2) = (0, 0)$, evaluate your utility as a function of m_1 and m_2. Comment on the interpretation of the constant that is specified in your utility function.

4. In a particular game, R chooses strategy R1, R2 or R3, C chooses strategy C1, C2, C3, C4 or C5. The payoffs to R are as follows

	C1	C2	C3	C4	C5
R1	2	0	3	-1	-2
R2	-2	3	0	0	4
R3	1	-2	4	-1	-2

The payoff to C is minus the payoff to R.

(a) Reduce this game to a game where R has only two possible strategies. Explain carefully why this can be done.

(b) Use a graphical method to identify the minimax strategies for R and for C, and the value of the game.

5. (a) Consider a bargaining problem with 4 options: A, B, C, D. The utilities for these options to John and David are given in the table below, together with their utilities for the status quo (SQ).

	A	B	C	D	SQ
John	-2	2	4	8	0
David	9	5	3	-1	1

i. Without optimising the function in the definition of the Nash point, show that at the unique solution to the Nash axioms John has utility 3 and David has utility 4. Explain briefly how this solution is derived and which Nash axioms are used. (*Hint:* Use a transformation to get a symmetric bargaining problem.)

ii. Specify *all* bargains over the options that correspond to the solution.

iii. Discuss whether or not a player may end up, at the end of the whole process of solving this decision problem, with an option for which he has lower utility than for the status quo; include the Individual Rationality axiom in your discussion.

(b) Consider a group decision problem with five voters and three alternatives, A, B and C. Each voter's preference ordering is transitive. Three voters prefer A over B and B over C. The other two voters prefer B over C and C over A. Combine these preferences into a group preference order using the Borda count procedure, assigning scores 2, 1 and 0, to a person's first, second and third preferred alternative, respectively. Discuss the resulting group preference order by considering all pairwise preferences based on the use of the simple majority rule.

6. Four lecturers (A, B, C, D) need to decide on the topic of a new module for the university degree programme. The options for the new modules are Statistics (St), Decision Theory (DT), Operations Research (OR) and Mathematical Finance (MF), while it is also possible not to introduce a new module (No). The utilities of the lecturers for these options are given in the table below.

	St	DT	OR	MF	No
A	10	8	7	0	0
B	0	1	1	2	0
C	4	3	0	1	0
D	0	1	0.5	0.5	0

(a) Apply Harsanyi's theorem of utilitarianism to rank these options. Briefly explain why using this theory may be attractive to solve this problem.

(b) Suppose the lecturers wish to explore an alternative solution method, based on the idea that they should choose the topic which minimizes the maximal 'unhappyness' of any single lecturer. Explain whether or not this can be implemented: If it can be done, give the solution; if it cannot be done, explain why not.

SECTION B

7. A market trader is considering whether to buy a certain quantity of a particular stock (decision d_1) or not (decision d_2).

   If decision d_1 is taken, then, given what she knows, the stock will go up (event S_1) with probability 0.5 or will go down (event S_2) with probability 0.5. If the stock goes up, profits will be £90,000, while if the stock goes down, the loss will be £60,000. If decision d_2 is taken, then there will be no profit or loss.

   Before deciding whether to buy the stock, she can use some computer time to model the market (m_1), or not (m_2). The computation will either make a positive prediction (event A_1) or a negative prediction (event A_2). The computation is judged to be 80% reliable, meaning that

$$P(A_1 \mid S_1) = P(A_2 \mid S_2) = 0.8 \ .$$

   (a) Find the probability of good sales, conditional on the computation making a *positive* prediction.

   Similarly, find the probability of good sales, conditional on the computation making a *negative* prediction.

   (b) Draw and solve the decision tree for the above problem, assuming that the computation is free and that the trader wishes to maximize expected money value.

   (c) Find and interpret:

   i. the expected value of the information to be provided by the computation, i.e. how much is the computation worth?

   ii. the expected value of perfect information about the stock price change, assuming that the computation is free.

   (d) Find the risk profile of the optimal solution obtained in part (b). Discuss what further considerations would be relevant to help the trader decide whether the best decision has been made.

8. We wish to estimate the parameter $\lambda \in \mathbb{R}_{>0}$ of a Poisson distribution. The prior distribution for λ is an exponential distribution with parameter b. The loss function for estimate d and value λ is

$$L(\lambda, d) = (\lambda - d)^2 + d^2 \ .$$

(a) Show that, before any data is gathered, the Bayes rule is $d^* = 1/2b$, and find the Bayes risk.

(b) Suppose that we may take a sample of n observations $\{X_i\}_{i\in[1..n]}$, from the Poisson distribution before estimating λ. Show that the posterior distribution for λ if we observe $\{X_i = x_i\}_{i\in[1..n]}$, is a gamma distribution.

(c) Find the Bayes rule, when we have observed $\{X_i = x_i\}_{i\in[1..n]}$, and show that the Bayes risk is

$$\frac{(n\bar{x}+1)^2 + 2(n\bar{x}+1)}{2(n+b)^2} \ ,$$

where $\bar{x}$ is the sample average.

(d) Find the Bayes risk of the sampling procedure, for a given sample size n.

(e) Suppose that the cost of a sample $\{X_i = x_i\}_{i\in[1..n]}$ is $\sum_i x_i$. Are there values of b for which it is worth taking a sample?

- The Poisson probability distribution on $k \in \mathbb{N}$, with parameter λ, is given by

$$P(k \mid \lambda) = e^{-\lambda}\lambda^k/k! \ .$$

- The exponential probability distribution on $\mathbb{R}_{>0}$, with parameter b, has pdf

$$p(x \mid b) = be^{-bx} \ .$$

- The gamma distribution on $\mathbb{R}_{>0}$,with parameters α and β, has pdf

$$p(y \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \ .$$

9. Amy and Lucy are planning to celebrate the end of their university exams by going away for a weekend. The destinations they consider are Amsterdam, Barcelona, London and Paris.

   Their utilities for each option are given in the following table.

	Amsterdam	Barcelona	London	Paris
Amy	2	4	5	7
Lucy	6	5	3	2

   As status quo option they consider not to go, for which both have utility 0.

   (a) Sketch the feasible region for this bargaining problem, identify the Pareto Boundary and the status quo point.

   (b) Find the Nash point and the equitable distribution point for this problem. What do you recommend Amy and Lucy to do?

   (c) Both the Nash point and the equitable distribution point satisfy the Pareto Optimality axiom. Formulate this axiom and explain briefly the role this axiom plays in these bargaining theories.

   (d) Suppose that before Lucy is asked to state her utilities for the four destinations, she learns Amy's utilities for them. Explain in detail whether or not she can use this information to manipulate the outcome of the bargaining problem, according to the Nash point, to her advantage.

10. (a) In a particular game, R chooses strategy R1 or R2, C chooses strategy C1, C2 or C3. The payoffs to R are as given in the table below, the payoff to C is minus the payoff to R.

	C1	C2	C3
R1	-2	1	-1
R2	4	-2	0

i. Identify the minimax strategies for R and for C, and the value of the game.

ii. State the minimax theorem for two-person zero-sum games and introduce the relevant concepts. Explain why a player may want to play their minimax strategy for such a game.

(b) The Hawk-Dove game has been presented in the lectures with pay-off table (with $W > 0$ and $L > 0$)

	H	D
H	$(\frac{W-L}{2}, \frac{W-L}{2})$	$(W, 0)$
D	$(0, W)$	$(\frac{W}{2}, \frac{W}{2})$

i. State the definition of the Nash Equilibrium for a two-person non-zero-sum game.

ii. Analyse the scenario $W \geq L$ and explain whether or not playing a strategy corresponding to a Nash Equilibrium is always a sensible strategy in a non-zero-sum game.

iii. Analyse the scenario $W < L$. Derive a strategy which might be played in this scenario if a player wishes to eliminate possible advantages the opponent may have, and briefly discuss the practical relevance of such a strategy.

END