

EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH3081-WE01

Title:

Numerical Differential Equations III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use diction	onaries: No	

the best FOUR answers from Section A and the best THREE answers from Section B. Questions in Section B carry TWICE as many marks as those in Section A.	
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Revision:

SECTION A

- 1. Consider the initial value problem $\dot{x} = f(x, t)$, $x(t_0) = x_0$, $t > t_0$.
 - (a) Define the local truncation error for a multi-step numerical scheme.
 - (b) Using finite difference, derive the Euler Backward method.
 - (c) Let $|\ddot{x}|$ be bounded. Show that the order of the Euler Backward method is 1.
- 2. Consider a numerical method

$$x_{n+1} = x_n + k \left[(1 - \theta) f(x_n, t_n) + \theta f(x_{n+1}, t_{n+1}) \right],$$

where $0 < \theta < 1$, applied to an initial value problem $\dot{x} = f(x, t)$, $x(t_0) = x_0$, $t > t_0$.

- (a) Explain whether the above method is:
 - i. explicit or implicit,
 - ii. single-step or multi-step.
- (b) Give the general form of a method of the Runge-Kutta family.
- (c) Show that the above method can be written in the form of a Runge-Kutta method.
- (d) Write down the Butcher tableau corresponding to this method.
- 3. Consider the numerical scheme

$$x_{n+2} = 3x_{n+1} - 2x_n - kf_n$$

applied to the initial value problem $\dot{x} = f(x, t)$, $x(t_0) = x_0$, $t > t_0$.

- (a) Write down the characteristic polynomials.
- (b) Give the definition of a consistent numerical scheme and show that the above scheme is consistent.
- (c) Give the definition of a zero-stable numerical scheme and determine whether the above scheme is zero-stable.





4. Let u = u(x) solve the following boundary-value problem

$$u''(x) + (x+1)u(x) = x^2 + 1$$
 for $x \in [0,1]$,
 $u'(0) - 2u(0) = 1$, $u'(1) = 0$.

- (a) Assuming that $u \in C^3([0, 1])$, write down the Taylor expansion for u(h) centred at 0 with accuracy $O(h^3)$, taking into account that u satisfies the differential equation.
- (b) Based on (a), find a second-order approximation for the boundary condition u'(0) 2u(0) = 1 in terms of u(0) and u(h). Justify your approximation.
- 5. For the equation $\partial_t u(x,t) + a \partial_x u(x,t) = 0$ (a is a real constant), consider the following approximation scheme

$$\frac{u_m^{n+1} - u_m^n}{\tau} + a \frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = 0 \quad \text{with } n \in \mathbb{Z}_+, \ m \in \mathbb{Z}.$$
 (1)

- (a) Considering a particular solution $u_m^n = \lambda(\phi)^n e^{im\phi}$, where $\phi \in [0, 2\pi]$, to the problem (1), find the amplification factor $\lambda(\phi)$.
- (b) Assuming that $\tau = rh/a$ for some constant r and using the spectral stability test, determine under which conditions on the parameter r the scheme (1) is stable.
- 6. (a) Consider the iteration

$$x^{k+1} = Gx^k + c$$

where G is a square matrix, and x^{j} and c are vectors, with x^{0} given. Assuming that the equation x = Gx + c has a unique solution, state necessary and sufficient conditions on G that guarantee the convergence of the iteration.

(b) Given a real matrix

$$A = \begin{pmatrix} \alpha & \beta & 0\\ 0 & \alpha & \beta\\ \beta & 0 & \alpha \end{pmatrix} \quad \text{with } \alpha \neq 0, \tag{2}$$

and some vector $b \in \mathbb{R}^3$, we seek to solve Ax = b using the Jacobi iteration process. Find the transition matrix G corresponding to A.

(c) Find all values of parameters α and β such that the Jacobi iteration process converges for arbitrary initial vector x^0 .



SECTION B

7. Consider the initial value problem

$$\ddot{x}+\frac{3}{2}\dot{x}=x\,,\,\,x(0)=0\,,\,\,\dot{x}(0)=1\,,\,\,t>0.$$

(a) By introducing new variables $x_1 = x$ and $x_2 = \dot{x}$ write the system in the form of $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,,$$

and A is a square matrix and hence find the analytical solution of the initial value problem.

- (b) Find an explicit formula for the approximate solution \mathbf{x}_n , produced by the Euler Forward method with step-size k > 0.
- (c) Show that the solution derived by the Euler Forward method converges to the analytical solution for $k \to 0$, in a fixed finite interval $t \in [0, T]$.
- 8. Consider an initial value problem $\dot{x}(t) = f(x(t), t)$, $x(0) = x_0$, t > 0. Let p_s be a polynomial of degree s interpolating f(x(t), t) at s + 1 nodes $t_i = ik$, (i = 0, ..., s):

$$p_s(t_i) = f(x(t_i), t_i), \quad i \in \{0, \dots, s\}.$$

This polynomial can be written in the Lagrange form

$$p_s(t) = \sum_{i=0}^s f(x(t_i), t_i)\ell_i(t), \text{ where } \ell_i(t) = \prod_{\substack{j=0\\j\neq i}}^s \frac{t-t_j}{t_i - t_j}.$$
(3)

(a) Use equation (3) along with a suitable change of variables to show that the *s*-step Adams-Bashforth method is

$$x_{n+1} = x_n + k \sum_{i=0}^{s-1} b_i f_{n-i}$$

where

$$b_i = \int_0^1 \prod_{\substack{j=0 \ j \neq i}}^{s-1} \frac{\tau+j}{j-i} d\tau$$
.

- (b) Derive the two-step Adams-Bashforth method.
- (c) Show that the s-step Adams-Bashforth method has order s.

- 9. Let $u = u(x) \in C^1([0, 1])$ be a real-valued function.
 - (a) Assuming that u(0) = u(1) = 0, state the continuous version of the Poincare inequality for u.
 - (b) Prove that

$$\sup_{x \in [0,1]} |u(x)| \le \int_0^1 |u'(y)| \, dy + \left| \int_0^1 u(y) \, dy \right|.$$

The assumption u(0) = u(1) = 0 is not needed here. Hint: consider $|u(x) - \int_0^1 u(y) \, dy|$.

Let $u_i, i \in \{0, \ldots, N\}$ be a real valued grid function.

- (c) Assuming that $u_0 = u_N = 0$, state the discrete version of the Poincare inequality for $u_i, i \in \{0, \ldots, N\}$.
- (d) Prove a discrete analogue of (b), namely

$$\max_{i \in \{0, \dots, N\}} |u_i| \le \left| \sum_{k=1}^{N-1} hu_k \right| + \sum_{k=1}^N |u_k - u_{k-1}|,$$

where hN = 1. The assumption $u_0 = u_N = 0$ is not needed here.

10. Let $N \in \mathbb{N}, N \geq 2$. Consider the following eigenvalue problem for a difference scheme

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} = -\lambda u_k \quad \text{for } 1 \le k \le N - 1 \quad \text{with } h(N-1) = 1, \qquad (4)$$
$$u_0 = -u_1, \quad u_N = -u_{N-1}. \tag{5}$$

- (a) Find the roots μ_1 and μ_2 of the characteristic polynomial associated to the difference equation (4). Find the values of $\mu_1 + \mu_2$ and $\mu_1 \mu_2$.
- (b) Under the assumption $\mu_1 = \mu_2 := \mu$, the general solution to the difference equation (4) takes the form

$$u_k = c_1 \mu^k + c_2 k \mu^k.$$

Taking into account the boundary conditions (5), find all λ such that the problem (4)–(5) has a nonzero solution.

(c) Under the assumption $\mu_1 \neq \mu_2$, the general solution to the difference scheme takes the form

$$u_k = c_1 \mu_1^k + c_2 \mu_2^k.$$

Taking into account the boundary conditions (5), find all λ such that problem (4)–(5) has a nonzero solution.