Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3091-WE01

Title:

Dynamical Systems III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. A two-dimensional dynamical system obeys the differential equations $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix}$$

and $\boldsymbol{x}$ is a two-dimensional state vector.

(a) Find a similarity transformation $A' = M^{-1}AM$ that puts A' into Jordan normal form.

(b) Use the result of part (a) to determine the general solution $\boldsymbol{x}(t)$ of the above linear system.

(c) Draw the corresponding phase flow.

2. The two-dimensional dynamical system

$$\dot{x} = y + x^2 \ , \quad \dot{y} = -x + y^2$$

is a non-linear deformation of the simple harmonic oscillator ($\dot{x} = y$, $\dot{y} = -x$).

(a) Determine the critical points of this system and find the linearised equations around each of them.

(b) Solve the linearised equations around each critical point and determine the centre, stable and unstable manifolds that they exhibit. Are any of the fixed points hyperbolic?

(c) Combining the information of the previous parts sketch a potential global phase flow for the full two-dimensional dynamical system.

3. A two-dimensional dynamical system has the form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + e^{At}\boldsymbol{G}(e^{-At}\boldsymbol{x})$$

where A is a 2×2 matrix and $\boldsymbol{x}(t)$, $\boldsymbol{G}(\boldsymbol{x})$ are vectors in $\mathbb{R}^2$. Let us denote the components of $\boldsymbol{x}(t)$ as $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(a) As a special case, assume

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \quad \boldsymbol{G}(\boldsymbol{x}) = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \ .$$

Determine the exponential matrices e^{At} and e^{-At}.

(b) Use the assumptions and results of part (a) to find the explicit form of the dynamical system equations for the functions $x_1(t), x_2(t)$. Is the resulting system autonomous?

(c) Show that for general A and $\boldsymbol{G}$ the dynamical system becomes autonomous if we set $\boldsymbol{x} = e^{At}\boldsymbol{y}$. In the special case of part (a) solve the autonomous system in the $\boldsymbol{y}$ coordinates and determine the general solution $\boldsymbol{x}(t)$.

4. Consider the one-dimensional dynamical system $\dot{x} = 1 - \mu x + x^2$.

(a) Determine the points (x_*, μ_*) where the system undergoes local bifurcation.

(b) Sketch the bifurcation diagram and identify the type of bifurcation.

(c) Give the normal forms of a one-dimensional dynamical system with saddle-node, transcritical and pitchfork bifurcation.

5. Consider the three-dimensional dynamical system

$$\dot{x} = y - x \ , \quad \dot{y} = 3x - y - xz \ , \quad \dot{z} = -x^2 - y^2 + 2z \ .$$

(a) Is this system Hamiltonian? Give your reasoning.

(b) Determine the fixed points of the system.

(c) Use Liouville's theorem to evaluate the rate of change of the volume $\mathrm{Vol}(\phi(t, D))$ for the domain $\phi(t, D)$ obtained by evolving all points in a generic domain D for some time t according to the above dynamical system.

(d) Prove that none of the fixed points obtained in part (a) is asymptotically stable.

6. Assume that an autonomous n-dimensional dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ with a C^1 function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ possesses a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ with the properties: (i) $\frac{dV}{dt} < 0$ on all $\mathbb{R}^n$ except for a bounded set S, (ii) $V(\boldsymbol{x}) > 0$ on all $\mathbb{R}^n$ except for S, (iii) $V(\boldsymbol{x}) = 0$ and $\frac{dV}{dt}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in S$.

(a) What does La Salle's principle imply about the late time behaviour of such a system?

Now consider the more specific two-dimensional dynamical system

$$\dot{x} = y + \frac{xe^x}{2}(1 - x^2 - y^2) \ , \quad \dot{y} = -x + \frac{ye^x}{2}(1 - x^2 - y^2) \ .$$

(b) Does the function $V(x, y) = (1 - x^2 - y^2)^2$ obey the above-mentioned assumptions? If so, what is the set S in this case?

(c) Using La Salle's principle, what can you conclude about the late time behaviour of the above two-dimensional dynamical system?

(d) What is the ω-limit set of the point $(x, y) = (0, \frac{1}{2})$?

SECTION B

7. A particle moves on the real line under the influence of a double-well potential. Its instantaneous position $x(t)$ obeys the second order ordinary differential equation (ODE)

$$\ddot{x} + 2(x^2 - 1)x = 0 \ .$$

   (a) Set $y = \dot{x}$ and re-express the second order ODE as a two-dimensional dynamical system. Show that this system is Hamiltonian. In other words, find a function $H(x, y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y} \ , \quad \dot{y} = -\frac{\partial H}{\partial x} \ .$$

   (b) Show that the function H that you found in part (a) is a first integral of the dynamical system. Use this fact to obtain an implicit algebraic equation for x, y that determines the trajectories of the system.

   (c) Determine the fixed points of the system and draw the complete phase portrait. Justify your answer using linear analysis around the fixed points and/or the implicit algebraic equation that you derived in part (b).

   (d) How many non-periodic trajectories exist in this system? Are they homoclinic, heteroclinic or none-of-the-above? What implicit algebraic equation of x, y describes these orbits?

   (e) For any of the non-periodic trajectories of part (d) determine the explicit form of the solution $x(t)$.
   [Useful integral: $\int dx \frac{1}{x\sqrt{2-x^2}} = \frac{1}{\sqrt{2}} \log \frac{x}{2+\sqrt{4-2x^2}}$].

8. Consider the two-dimensional dynamical systems

$$\dot{x} = y(x^2 - 1) \ , \quad \dot{y} = x(1 - y^2) \ . \tag{1}$$

   (a) Identify the critical points of this system. Specify which of these points are hyperbolic.

   (b) Show that the four lines $x = -1$, $x = 1$, $y = -1$, $y = 1$ are invariant sets and determine if these sets contain heteroclinic orbits. Identify these orbits if they exist.

   (c) Does the stable manifold theorem apply to all fixed points of the system? Summarise the statement of the stable manifold theorem and explain in this system where it does or does not apply, and why.

   (d) Are there periodic orbits in this system? If yes, in what region of the phase flow do you expect them?

   (e) Draw the phase portrait combining the information of the previous parts. Can you identify other invariant sets besides the ones in part (b)?

9. The two-dimensional non-linear dynamical system

$$\dot{x} = x - 2y + x^3(1 - x^2 - y^2) \ , \quad \dot{y} = x + y^3(1 - x^2 - y^2) \tag{2}$$

has a single fixed point at $(x, y) = (0, 0)$.

(a) At large distances from the origin, $r = \sqrt{x^2+y^2} \gg 1$, approximate the differential equations (2) by their leading order behaviour as

$$\dot{x} \simeq -x^3(x^2+y^2) \ , \quad \dot{y} \simeq -y^3(x^2+y^2) \ .$$

Compute the orbital derivative of the radial coordinate, $\dot{r}$, in this approximation on a circle of radius R and centre $(0,0)$ and show that it is strictly negative. Use this observation to argue that a disc $D = \{(x,y) \,|\, x^2+y^2 \leq R\}$ with sufficiently large radius $R \gg 1$ is a compact, positively invariant set of the original dynamical system (2).

(b) What does the ω-limit set theorem imply for the ω-limit set $\omega(p)$ of any point p in the disc D of part (a)?

(c) According to the Poincaré-Bendixson theorem, there are in principle several possible types of ω-limit set for any point inside the positively invariant disc D. What are these possibilities? Given the fact that the system (2) has only one fixed point at the origin, which of these possibilities remain?

(d) Linearise the system (2) around the origin and determine the nature of the fixed point at the origin. Can the origin be in $\omega(p)$ for any p which is not the origin? Combined with the answer in part (c) what can $\omega(p)$ be for any p in the disc D that is not the origin?

(e) State (but do not prove) Bendixson's criterion for a two-dimensional dynamical system.

(f) Apply Bendixson's criterion to the dynamical system (2) on the disc with radius $\frac{1}{2}$ centred at the origin. Can a limit cycle exist inside this disc?

10. Consider a two-dimensional dynamical system

$$\dot{x} = f(x,y) \ , \quad \dot{y} = g(x,y) \ .$$

(a) Let γ be a smooth closed curve in $\mathbb{R}^2$. We assume that f^2+g^2 is nowhere zero on γ. Give the definition of the Poincaré index of γ and list its values for curves that encircle a source, or a sink, or a centre, or a saddle fixed point respectively.

(b) Prove that a periodic orbit can only enclose an odd number of fixed points.

(c) A two-dimensional system is called a gradient system if there is a function $V(x,y)$ for which

$$\dot{x} = -\frac{\partial V}{\partial x} \ , \quad \dot{y} = -\frac{\partial V}{\partial y} \ .$$

Show that the orbital derivative of V has $\frac{dV}{dt} \leq 0$.

(d) Using the result of part (c) show that gradient systems cannot have periodic orbits.

(e) Is the two-dimensional dynamical system

$$\dot{x} = y + 2xy \ , \quad \dot{y} = x + x^2 - y^2 \qquad (3)$$

a gradient system? If yes, then what is the corresponding function $V(x,y)$?

(f) Determine the fixed points of the dynamical system (3) and explain if they are sources, sinks, saddles or centres.

(g) If you did not know about the result of part (d), would you be able to argue using the Poincaré index that there can be no periodic orbits in the system (3)? [Hint: examine what lessons can be drawn from the Poincaré index if the putative periodic orbit encircles no fixed points, each of the different fixed points, or two fixed points.]