Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3111-WE01

Title:

Quantum Mechanics III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Consider a Quantum Mechanical system with a two-dimensional Hilbert space spanned by the orthonormal basis $S = \{|a\rangle, |b\rangle\}$. An observable $\hat{M}$ is defined through

$$\hat{M}\,|a\rangle = |a\rangle + 2\;|b\rangle, \qquad \hat{M}\,|b\rangle = |b\rangle + 2\;|a\rangle\,.$$

   (a) Find the matrix form of $\hat{M}$ in the basis S.

   (b) What are the possible outcomes in a measurement of $\hat{M}$?

   (c) Write the state immediately after a measurement of $\hat{M}$ which yielded the smallest value possible.

2. Consider a two-dimensional Hilbert space and an observable $\hat{\Omega}$.

   (a) If $\hat{P}_1$ and $\hat{P}_2$ are the projection operators on the eigenspaces of $\hat{\Omega}$ and you are given that

$$\hat{P}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

   find the matrix form of $\hat{P}_2$.

   (b) If the corresponding eigenvalues are $\omega_1 = 1$ and $\omega_2 = -1$, find the matrix form of $\hat{\Omega}$.

   (c) Compute the expectation value $\langle\,\hat{\Omega}\,\rangle$ for the state

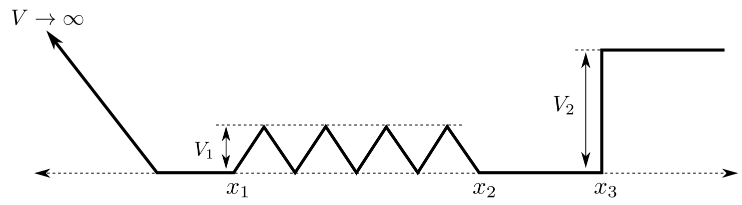
$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. (a) State how an observable $\hat{\Omega}$ transforms passively under a unitary transformation $\hat{U}$.

   (b) Consider a one-dimensional system with position and momentum operators $\hat{x}$ and $\hat{p}$. Under a unitary transformation $\hat{U}$, they transform to the new operators $\hat{x}'$ and $\hat{p}'$. Find the commutation relation that the new position and momentum operators satisfy.

   (c) For a two-dimensional system, the angular momentum operator is given by $\hat{L} = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$. Compute the commutators $\left[\hat{x}^2, \hat{L}\right]$, $\left[\hat{y}^2, \hat{L}\right]$ and $\left[\hat{x}^2 + \hat{y}^2, \hat{L}\right]$. What does that tell you about the way that $\hat{x}^2 + \hat{y}^2$ transforms under rotations?

4. The potentials for two different one-dimensional quantum systems are shown in the figure below.



A quantum particle is propagating in one dimension under the influence of these potentials. Answer the following questions:

   (a) Is the energy eigen spectrum of the particle in the potential A) discrete or not? If only part of the spectrum is discrete, explain when this happens.

   (b) Is the energy eigen spectrum of the particle in the potential B) discrete or not? If only part of the spectrum is discrete, explain for which values of energies this happens.

   (c) For the potential B) state for which values of the variable x the wave functions are oscillatory and for which values of x they are not. If the range in which this happens depends on the energy E of the particle, please clearly say for which values of x the behaviour of wave functions changes.

   (d) For the potential B), and for the particle of energy $E < V_2$, indicate where the probability of finding the particle is the smallest. In the limit $V_2 \to \infty$ and for the *highly excited* particle of energy E, with $V_1 \ll E < V_2$, indicate where the probability of finding the particle is the largest.

5. (a) Write down the basic commutation relation $[\hat{L}_i, \hat{L}_j]$ between arbitrary angular momentum operators $\hat{L}_i$ $(i, j = x, y, z)$.

   (b) Using the answer from the previous part, evaluate the following four commutators

$$[\hat{L}_x \hat{L}_y, \hat{L}_x]\,, \qquad [\hat{L}^2, \hat{L}_x + \hat{L}_y]\,, \quad [\hat{L}_x, \hat{x}]\,, \quad [\hat{L}_y, \hat{z}]\,,$$

   where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

6. A system consists of two particles of mass m which are attached to the ends of a massless *rigid* rod of length a. Assume that this system is free to rotate in three-dimensional space about its center, but that the center of the system is fixed.

   (a) Write the classical and quantum expressions for the energy of this system, and express them using the angular momentum of the system.

   (b) Evaluate the energy spectrum of this system and say what are the eigen energy states and what is the degeneracy of the n-th energy level.

SECTION B

7. Consider a Quantum Mechanical system with observables $\hat{s}_x$, $\hat{s}_y$ and $\hat{s}_z$ satisfying

$$[\hat{s}_x, \hat{s}_y] = i\,\hbar\,\hat{s}_z, \quad [\hat{s}_y, \hat{s}_z] = i\,\hbar\,\hat{s}_x, \quad [\hat{s}_x, \hat{s}_z] = -i\,\hbar\,\hat{s}_y\,.$$

The Hamiltonian operator is given by $\hat{H} = \mu\,\hat{s}_z$ with μ a real number.

(a) Show that if the Hamiltonian $\hat{H}$ of any Quantum Mechanical system is time independent, then the expectation value $\langle\,\hat{\Omega}\,\rangle$ of any observable $\hat{\Omega}$ satisfies

$$\frac{d}{dt}\langle\,\hat{\Omega}\,\rangle = \langle\,\frac{i}{\hbar}\left[\hat{H}, \hat{\Omega}\right]\,\rangle\,.$$

(b) For the system at hand, determine the time dependence of the expectation values $\langle\,\hat{s}_x\,\rangle$, $\langle\,\hat{s}_y\,\rangle$ and $\langle\,\hat{s}_z\,\rangle$. Your final answer should be parametrised by three constants of integration and the number μ.
Hint: You are given that the most general real solution of the system $\dot{a}(t) = \omega\,b(t)$, $\dot{b}(t) = -\omega\,a(t)$ has $a(t) = A\,\cos(\omega\,t) + B\,\sin(\omega\,t)$ with A and B real constants.

(c) You are now given that the Hilbert space of the system you have been studying is two-dimensional and that

$$\hat{s}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{s}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Determine the matrix form of $\hat{s}_z$ and fix the three constants of integration you were left with in the previous question for the initial state

$$|\,\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d) For the system and initial state you were given in c), find the times t_n at which the time-evolved state $|\,\psi(t)\rangle$ becomes an eigenstate of $\hat{s}_x$.

8. Consider the isotropic two-dimensional simple harmonic oscillator of frequency ω and mass m. The Quantum Mechanical Hamiltonian $\hat{H}$ and angular momentum $\hat{L}$ operators can be expressed as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1\right)$$
$$\hat{L} = \hbar \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\right)$$

with the only non-trivial commutators among the operators $\hat{a}^\dagger$, $\hat{a}$, $\hat{b}^\dagger$ and $\hat{b}$ being

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1\,.$$

We define the unit norm states

$$|\, m, n\rangle = \frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^m\,(\hat{b}^\dagger)^n\;|\,0,0\rangle$$

for m and n non-negative integers and with $|\,0,0\rangle$ being the correctly normalised ground state satisfying $\hat{a}\,|\,0,0\rangle = \hat{b}\,|\,0,0\rangle = 0$.

(a) Use induction to show that $[\hat{a}, (\hat{a}^\dagger)^m] = m\,(\hat{a}^\dagger)^{m-1}$ and $[\hat{b}, (\hat{b}^\dagger)^n] = n\,(\hat{b}^\dagger)^{n-1}$ with m and n being positive integers.

(b) Compute the commutator between $\hat{H}$ and $\hat{L}$. Is angular momentum conserved? Is the Hamiltonian invariant under small rotations? Justify your answers.

(c) Show that the states $|\,m, n\rangle$ are simultaneous eigenstates of $\hat{H}$ and $\hat{L}$. Give the expressions for the corresponding eigenvalues in terms of m and n.

(d) Consider a state $|\,\psi\rangle$ for which experimentalists measure $\hat{H}$ and $\hat{L}$. For $\hat{H}$ they only find the outcomes $\hbar\omega$ and $2\hbar\omega$ with equal probability. For $\hat{L}$ they only find the values 0 and $-\hbar$. Write down the most general state that is consistent with these observations. Given these constraints, what are the probabilities to measure angular momentum equal to 0 ?
Hint: You should end up with a one parameter family of states.

(e) Consider the state $|\,\psi\rangle$ that you determined in the previous question as the initial state and find how it evolves with time t.

9. The exact two-dimensional simple harmonic oscillator (SHO) in one dimension has a potential given by

$$\hat{V}_0 = \frac{1}{2}\kappa^2(x^2 + y^2)\,,$$

and is perturbed by the Hamiltonian

$$\hat{H}' = \alpha(\hat{a}_x^\dagger + \hat{a}_x a_y^\dagger)$$

where $\alpha > 0$ is a small constant, and $\hat{a}_x\hat{a}_y, \hat{a}_x^\dagger, \hat{a}_y^\dagger$ are annihilation and creation operators for the one-dimensional simple harmonic oscillators along the x and y axis.

(a) Write down the spectrum of the unperturbed system ($\alpha = 0$) and explain what is the degeneracy of each eigen energy state.

(b) Apply first order perturbation theory to compute the energy shift to the ground state, when $\alpha \neq 0$.

(c) Apply first order perturbation theory to compute the corrections to the ground state of the free system, when $\alpha \neq 0$.

(d) Apply first order perturbation theory to compute corrections to the energy of the second excited state.

Hint: Recall from the lecture that the second excited state has the total occupation number (i.e. the total number of excitations) $n = 2$, and don't worry that the perturbation Hamiltonian is not Hermitian.

10. In the WKB approximation the wave function of the quantum particle in the potential $V(x)$, is given (away from the turning points) by a linear combination of the wave functions

$$\Psi_{WKB,\pm}(x) = \frac{a_\pm}{\sqrt{|p(x)|}} e^{\pm\frac{1}{\hbar}\int_0^x p(y)dy}\,, \quad p(x) = \sqrt{2m/\hbar(V(x)-E)}\,,$$

where $V(x)$ can be larger or smaller than E.

For $V(x)$ we will consider two possibilities, $V_A(x)$ or $V_B(x)$, given by

$$V_A(x) = \begin{cases} \infty & x<0 \\ 0 & 0<x<L \\ V_0 x & L<x<2L \\ \infty & x>2L \end{cases} \qquad V_B(x) = \begin{cases} 0 & x<0 \\ V_0+\alpha x & 0<x<L \\ 0 & L<x \end{cases},$$

where $V_0 > 0$ is a positive constant. One quantum particle of mass m is bouncing inside the potential $V_A(x)$ and another quantum particle, also of mass m, is approaching the potential $V_B(x)$ from $x=-\infty$.

Apply the WKB approximation to answer the following questions:

(a) For the first system with potential $V_A(x)$, write down the boundary conditions which the wave function of the particle has to satisfy at $x=0$ and $x=2L$.

(b) What is the energy spectrum for the particle of energy $E > 2V_0L$ inside the potential $V_A(x)$? You do not have to solve explicitly for E_n, but you can leave the result at the level of the equation which determines the energy.

(c) Following the lectures, explain the general logic for computing the tunneling probability for a particle which approaches the potential $V_B(x)$, in the WKB approximation.

(d) Apply the general logic from the previous part of the question to evaluate what is the probability for a particle which approaches the potential $V_B(x)$ from the left with the energy $E < V_0$ to tunnel through this potential.