



EXAMINATION PAPER

Examination Session: May	Year: 2019	Exam Code: MATH3141-WE01
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Title: Operations Research

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section A and the best THREE answers from Section B. Questions in Section B carry TWICE as many marks as those in Section A.
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Revision:	
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SECTION A

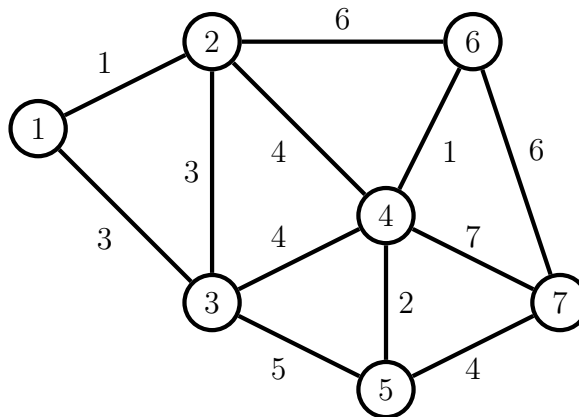
1. Solve the following linear program by using the two phase method, and state all feasible values of x_1 , x_2 , and x_3 for which the optimal value is attained.

$$\begin{aligned} \max \quad & -3x_1 + 4x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_3 \leq 4 \\ & x_1 - x_2 - x_3 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{aligned}$$

2. Consider the transportation problem with costs, supplies, and demands respectively given by:

$$[c_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 6 \end{bmatrix} \quad [a_i] = \begin{bmatrix} 30 \\ 70 \end{bmatrix} \quad [b_j] = [60 \quad 20 \quad 20]$$

- (a) Find the optimal transportation scheme.
 - (b) Consider the same problem but where transportation from source 1 to destination 1 (with $c_{11} = 1$) is no longer possible. Derive the new optimal transportation scheme.
3. Consider the following network:



The numbers on the arcs denote distances between nodes.

- (a) Apply Dijkstra's algorithm to find the shortest distance from node 1 to node 7.
- (b) Identify all shortest paths from node 1 to node 7.

4. An allotment enthusiast wishes to plant their 17 foot wide allotment space with rows of carrots, potatoes and pumpkins. Each row of carrots takes up 2 feet of space, each row of potatoes takes up 4 feet of space while each row of pumpkins takes up 7 feet of space. The allotment enthusiast's personal returns (on a scale of 1 to 10) are 1 for each row of carrots, 4 for each row of potatoes and 9 for each row of pumpkins. However, the allotment enthusiast has already 2 rows of potatoes planted from last winter, which they are not willing to dig out (so there are already 2 rows of potatoes planted). Formulate the above resource allocation problem and use dynamic programming to find the selection of rows that maximizes the enthusiast's total personal reward.
5. Consider the economic lot-size model in which shortages are not permitted, the rate of continuous demand is a , orders have setup cost K and unit cost c , and the continuous holding cost is h per unit of inventory per unit time.

(a) Show that the average cost per unit time as a function of the order size Q is

$$U(Q) = \frac{aK}{Q} + ac + \frac{Qh}{2}.$$

(b) Deduce a formula for the optimal order size Q^* in terms of a , K , and h .

Suppose now that a 50% quantity discount is introduced, so that the unit cost is reduced to $c/2$ on orders of at least 15 units.

(c) If $K = 100$, $c = 1$, $a = 2$, and $h = 1$, find the optimal order size for the model with 50% quantity discount.

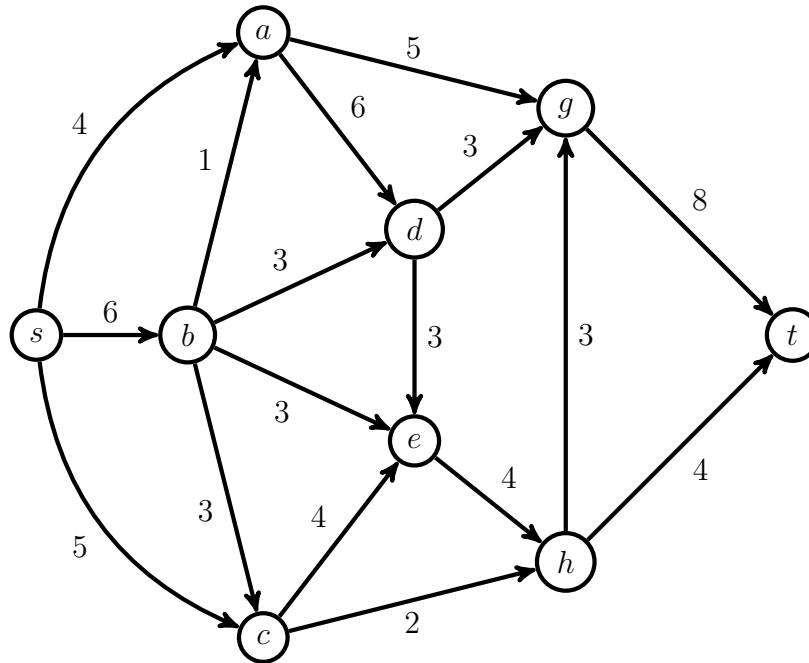
6. Consider a Markov chain (X_n) with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- (a) What is the probability of hitting state 6 before state 1 when starting from state 2?
- (b) Compute the expected time of reaching state 1 when starting from state 2.

SECTION B

7. Consider a transportation system that can be represented by the following flow network, with capacities as labelled:



We are specifically interested in the flow from s to t .

- Apply the Ford-Fulkerson algorithm to find the maximum flow from the source s to the terminus t .
- Find a cut separating s and t with minimal capacity.
- Identify all arcs that provide an opportunity for increasing the maximal flow, and provide a theoretical argument as to why only these arcs provide this opportunity. You may use, without proof, any theorems proved in the lectures, as long as you state them clearly.
- Pick one of the arcs that you just identified. Increase its capacity by 3, and find the new maximal flow as well as a new cut between s and t with minimal capacity.

8. Consider a case where someone has solved the following linear programming problem:

$$\max x_1 + 3x_2 + 4x_3 + 2x_4 \quad (1a)$$

$$\text{subject to } 2x_1 + x_2 + x_3 + 2x_4 \leq 2 \quad (1b)$$

$$3x_1 + x_2 + 2x_3 + 2x_4 \leq 3 \quad (1c)$$

$$x_1 + 2x_2 + 4x_3 + x_4 \leq 7 \quad (1d)$$

and subject to all $x_i \geq 0$ for $i \in \{1, 2, 3, 4\}$. The initial simplex table for this problem is

T_0	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
z	-1	-3	-4	-2	0	0	0	0
s_1	2	1	1	2	1	0	0	2
s_2	3	1	2	2	0	1	0	3
s_3	1	2	4	1	0	0	1	7

(2)

and the final simplex table is

T_*	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
z	6	0	0	4	2	1	0	7
x_2	1	1	0	2	2	-1	0	1
x_3	1	0	1	0	-1	1	0	1
s_3	-5	0	0	-3	0	-2	1	1

(3)

Now consider a similar linear programming problem, with modifications as indicated in bold:

$$\max x_1 + 3x_2 + 4x_3 + 2x_4 \quad (4a)$$

$$\text{subject to } 2x_1 + x_2 + x_3 + 2x_4 \leq 2 \quad (4b)$$

$$3x_1 + x_2 + 2x_3 + 2x_4 \leq 3 - \mathbf{2\alpha} \quad (4c)$$

$$x_1 + 2x_2 + 4x_3 + x_4 \leq 7 - \mathbf{6\alpha} \quad (4d)$$

and still subject to all $x_i \geq 0$ for $i \in \{1, 2, 3, 4\}$. Here, α is an arbitrary fixed value in $\mathbb{R}$. For this modified problem, use post-optimal analysis to answer all of the following questions:

- (a) For what values of $\alpha \in \mathbb{R}$ does the basis $\{x_2, x_3, s_3\}$ give a basic feasible solution?
- (b) For what values of $\alpha \in \mathbb{R}$ does the basis $\{x_2, x_3, s_3\}$ give an optimal basic feasible solution? In those cases where the basis $\{x_2, x_3, s_3\}$ gives an optimal basic feasible solution, what is the optimal value of the objective function as a function of α ?
- (c) Find the optimal solution of the modified problem for $\alpha = 1$.

9. The newly developed tutorial centre in Durham's mathematical sciences department consists of a sequence of service periods. A student that completes the service in a period departs at the end of that period. Just at the end of each service period, a new student arrives with probability $1/2$. The next service period starts soon after. If there are already two students in the centre, the newly arriving student gets frustrated, leaving immediately and vows never to return again. However, if there is one student or less already in the centre, the newly arriving student will join the service. At most one new student who can join the service arrives per period and at most one student can be served in a period.

The manager of the tutorial centre has two service configurations available: deploy a postgraduate fellow, labelled a PG configuration, or deploy a teaching fellow, labelled a TF configuration. The manager decides which configuration to use at the beginning of each period (so service completions precede arrivals which precede decisions). The PG configuration costs 3 units and the student being served during the period will complete the service and leave the centre at the end of the period with probability $3/5$ (otherwise, the student remains in the system). The TF configuration costs 9 units and the student being served during that period will complete and leave the tutorial centre with probability $4/5$. The tutorial centre earns a credit of 50 units each time a student leaves the service.

- (a) Formulate the problem of choosing the service configuration, period by period, as a Markov decision process. Identify the states and decisions. For each combination of states and decisions, find the *expected net immediate return* (subtracting any running costs) earned during that period.
- (b) The manager is currently implementing policy A_1 , where the PG configuration is used in all states. Perform two complete steps of the policy improvement algorithm to find an improved policy for the long-run expected reward per unit time. After two steps of the policy improvement algorithm, are you able to conclude whether you have reached the optimal policy or not?

10. Three investment schemes (A , B , and C) accept investments in multiples of £3000. Schemes A and B mature after one year, and generate returns distributed as follows:

	annual return on £3000 investment	probability
A	0	0.2
	5000	0.8
B	3000	0.4
	4000	0.3
	5000	0.3

Tabulated returns include the original investment so that, for example, 0 means loss of the entire investment and investment B has a 40% chance of breaking even over one year.

Scheme C matures after two years, generating a guaranteed return as follows:

	two-year return on £3000 investment	probability
C	5000	1.0

You have £3000 to invest at the start of a period of N years.

At the end of a year that does not bring the investment period to a close, any return from that year, together with any amount not invested that year, is available to invest for the next year. Investment may be split between the schemes as desired (provided the rule that investments must be in units of £3000 is respected).

Use stochastic dynamic programming to find the optimal (in terms of expected total profit) investment strategy for the *first year* of the given investment period in the case where

- (a) $N = 1$;
- (b) $N = 2$;
- (c) $N = 3$.

Hint: Although you can re-use many calculations from one part of this question to the next, be careful to take full account of money left over from previous years, particularly in answering part (c).