EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH3281-WE01

Title:
Topology III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

Use a separate answer book for this Section.

1. (a) Give the definition of a topological space. Explain what it means for a topological space to be Hausdorff.

   (b) Show that if X is a Hausdorff topological space and $p \in X$ then $\{p\}$ is closed.

   (c) Give an example of non-Hausdorff topological space X in which $\{p\}$ is closed whenever $p \in X$.

2. (a) State what it means for a topological space to be compact.

   (b) Let $f : X \to Y$ be a continuous map between topological spaces X and Y, and suppose that X is compact. Show that $f(X) \subseteq Y$ is compact in the subspace topology.

3. (a) Suppose X and Y are topological spaces and $f : X \to Y$ is a function. State what it means for f to be continuous.

   (b) Suppose $X = \{1, 2, 3\}$ and $Y = \{A, B\}$ are topological spaces with topologies τ_X and τ_Y given by

   $$\tau_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X\}$$

   and

   $$\tau_Y = \{\emptyset, \{A\}, Y\}.$$

   Find all functions $f : Y \to X$ which are continuous.

4. (a) State the definition of the Euler characteristic of a closed surface in terms of graphs drawn on it, sketching the steps of the argument as to why the definition is independent of the graph chosen.

   (b) Let A and B be closed, triangulable surfaces, and denote by $A\#B$ their connected sum. State and prove a relationship between the Euler characteristics of A, B and $A\#B$.

5. (a) Let (X, x_0) be a pointed space. State the definition of the fundamental group $\pi_1(X, x_0)$, including the group operations.

   (b) Now assume X is a metric space, with metric $d\colon X \times X \to \mathbb{R}$. Define the metric space ΩX as the set whose elements are maps $f\colon [0, 1] \to X$ with $f(0) = f(1) = x_0$ and metric D given by

   $$D(f, g) \;=\; \mathrm{Sup}_{t\in[0,1]}\big\{d(f(t), g(t))\big\}\,.$$

   Prove that if $\pi_1(X, x_0)$ is the group with one element then ΩX is path connected.

6. Let Δ be the space defined as the quotient of the solid triangle with vertices A, B and C given by identifying the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ (in those directions). Give a triangulation of Δ and use it to compute the fundamental group $\pi_1(\Delta)$, briefly explaining your calculations.

SECTION B

Use a separate answer book for this Section.

7. You may assume in this question that path-connected spaces are connected.

   (a) Suppose X and Y are topological spaces with X connected and $f : X \to Y$ is continuous. Show that $f(X)$ is connected in the subspace topology.

   (b) Show that the connected components of $\{0\} \cup \{1/n : n = 1, 2, 3, \ldots\} \subseteq \mathbb{R}$ are those subsets containing single points.

   (c) Consider the set
   $$U = \{(x, y) \in \mathbb{R}^2 : x > 0\},$$
   and the (clearly continuous) function $m : U \to \mathbb{R}$ given by $m(x, y) = y/x$. For $n \geq 1$, let $L_n \subseteq \mathbb{R}^2$ be the closed straight line segment with endpoints $(0, 0)$ and $(1, 1/n)$. Consider
   $$C = \{(1, 0)\} \cup \bigcup_{n=1}^{\infty} L_n \subseteq \mathbb{R}^2.$$
   Using m or otherwise show that if $r > 0$ and
   $$p : [0, r] \to C \setminus \{(0, 0)\}$$
   is continuous with $p(0) = (1, 0)$ then $p([0, r]) = \{(1, 0)\}$.

   (d) Hence or otherwise show that C is not path-connected.

8. (a) Let $\mathbb{R}^+$ be the topological group of positive real numbers with the group operation being multiplication. Show that
   $$a \cdot (x, y, z) = (ax, ay, az)$$
   (where $a \in \mathbb{R}^+$ and $(x, y, z) \in \mathbb{R}^3$) defines a topological group action of $\mathbb{R}^+$ on $\mathbb{R}^3$.

   (b) Since the orbit of $(0, 0, 0)$ under this action just consists of itself, we may consider the action of $\mathbb{R}^+$ on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Show that the orbit space
   $$(\mathbb{R}^3 \setminus \{(0, 0, 0)\})/\mathbb{R}^+$$
   is homeomorphic to S^2.

   (c) Describe with justification all the open sets of $\mathbb{R}^3/\mathbb{R}^+$ which contain the orbit of $(0, 0, 0)$.

9. (a) In this question we shall consider S^1 to be the unit circle in the complex plane $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$, and we recall that $\pi_1(S^1) = \mathbb{Z}$. If $f\colon S^1 \to S^1$ is a map, define the *degree* of f, denoted $\deg(f)$, to be the number $f_*(1) \in \mathbb{Z} = \pi_1(S^1)$. Suppose g is also a map $S^1 \to S^1$. Compute the degree of the composite $f \circ g$ in terms of $\deg(f)$ and $\deg(g)$.

   (b) Let $a\colon S^1 \to S^1$ be the *antipodal map*, given by $a(z) = -z$. Compute $\deg(a)$ using any techniques from lectures you wish.

   (c) By giving explicit maps $\alpha\colon \mathbb{C} - \{0\} \to S^1$ and $\beta\colon S^1 \to \mathbb{C} - \{0\}$, prove that $\mathbb{C} - \{0\}$ is homotopy equivalent to S^1.

   (d) Given a map $\gamma\colon [0,1] \to \mathbb{C} - \{0\}$ with $\gamma(0) = \gamma(1)$, define the *winding number* $w(\gamma, 0)$. Use your maps α and/or β to state a relationship between the notions of winding number and degree, briefly justifying your answer.

   (e) Compute the winding number about the origin of the map $[0,1] \to \mathbb{C} - \{0\}$ given by $t \mapsto 4e^{4\pi i t} + e^{2\pi i t}$.

10. (a) Recall that a closed surface is always compact, connected and without boundary. State the classification theorem for closed surfaces.

   (b) Explain what is meant by a *non-separating closed curve* in a surface S. Explain the process of *surgery* along a non-separating closed curve, and what consequence this operation has to the Euler characteristic.

   (c) Suppose S and R are closed triangulated surfaces and $f\colon S \to R$ is a simplicial map with the property that for each n-simplex σ in R, there are exactly r n-simplices in S which map to σ, each simplex mapping homeomorphically to σ. We call such a surface S an *r-fold covering surface* of R. What can you say about the relation of the Euler characteristics of S and R?

   (d) Give examples of two (distinct) closed surfaces neither of which can have a 4-fold covering surface, using your answers to parts (b) and (c) to justify your answers.

   (e) Let S be a 2-fold covering surface of R, and $f\colon S \to R$ the associated map. Suppose R and S are both orientable and that γ is a non-separating simple closed curve in R. Let ψ be the set of points in S which map by f to the points of γ, and suppose ψ is a non-separating simple close curve in S. If S' and R' are the surfaces obtained by doing surgery along ψ and γ respectively, can S' still be a 2-fold covering surface of R'? Justify your answer.