EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May	2019	MATH3361-WE01

Title:

Topics in Statistics III

Time Allowed:	3 hours	
Additional Material provided:	Tables: Normal, t-distribution, F-distribution, χ^2-distribution; Graph paper. Mantel-Haenszel test statistic	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

SECTION A

1. All possible hierarchical log-linear models were fitted to a contingency table, based on a sample of 312 individuals, for three factors: X (2 levels), Y (3 levels) and Z (5 levels). The sampling scheme that was implemented was the Poisson sampling scheme. The results are shown in the following Table 1.

Model	Log-likelihood
$[X, Y, Z]$	-71.75
$[Z, XY]$	-71.69
$[Y, XZ]$	-70.37
$[X, YZ]$	-48.13
$[XY, XZ]$	-70.30
$[XY, YZ]$	-47.83
$[XZ, YZ]$	-39.83
$[XY, XZ, YZ]$	-39.30
$[XYZ]$	-38.00

Table 1: Results.

   (a) Calculate the number of free parameters resulting after applying corner point non-identifiability constraints for each model in the table.

   (b) Define the Akaike Information Criterion (AIC), and define the Bayes Information Criterion (BIC). Also, compute AIC and BIC for each model in the table.

   (c) Identify which model is selected by each criterion and give a short explanation of what each criterion is intended to achieve.

   (d) Explain precisely what model $[XY, YZ]$ says about the dependence of factor Y on the other two factors.

2. Consider an $I \times J \times K$ contingency table, with classification variables X, Y, Z. The cell probabilities of the table are π_{ijk} for $i = 1, ..., I$, $j = 1, ..., J$, and $k = 1, ..., K$. Prove that if

   - X and Y are conditionally independent on Z, and
   - X and Z are conditionally independent on Y,

   then Y and Z are jointly independent of X.

3. Let $x_1, x_2, ..., x_n$ be independent and identically distributed random samples generated from a distribution f_θ admitting some density $f(x|\theta)$ and labeled by an unknown parameter $\theta \in \mathbb{R}$. Consider that interest lies in testing the pair of hypotheses $\mathrm{H}_0 : \theta = \theta_*$ versus $\mathrm{H}_1 : \theta \neq \theta_*$, where $\theta_* \in \mathbb{R}$ is a known fixed value. For this reason, one constructs three hypothesis tests at significance level a, based on the likelihood ratio statistic, Wald's statistic, and the Score statistic. Each of these hypothesis tests has rejection areas of the form

$$\{x_{1:n} : \ W_n(\theta) \geq \chi^2_{1,1-a}\}$$

where $\chi^2_{1,1-a}$ denotes the $1-a$ quantile of the chi-square distribution with 1 degree of freedom, and $W_n(\theta)$ denotes a statistical function which may be different for each of the aforesaid tests.

(a) State $W_n(\theta)$ for each of the three rejection areas. Intuitively justify the inequality in the rejection area for each of the tests by using geometrical arguments. You can make use of a plot like the one below to present your explanations.

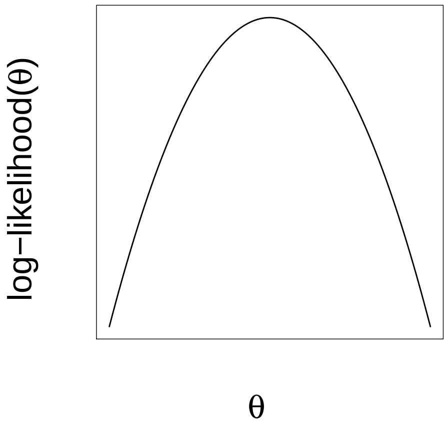


Figure 1: Figure for question 3(a)

(b) Assume that the sampling distribution f_θ is the logistic distribution admitting density

$$f(x|\theta) = \frac{\exp(-(x-\theta))}{(1+\exp(-(x-\theta)))^2}$$

labeled by an unknown parameter $\theta \in \mathbb{R}$. Find the rejection area of the form

$$\{x_{1:n} : \ W_n(\theta) \geq \chi^2_{1,1-a}\}$$

based on the Score statistic.

Hint: You may use the fact that

$$\int_{-\infty}^{\infty} \left(\frac{\exp(-(x-\theta))}{(1+\exp(-(x-\theta)))^2} \right)^2 \mathrm{d}x = \int_0^1 \xi(1-\xi)\mathrm{d}\xi$$

4. We are interested in researching factors associated with coronary heart disease in an area of the Western Cape of South Africa which has high heart disease risk. A sample of data relating to $n = 463$ males is available as a data frame `heart`, the first six rows of which are provided in the R excerpt at the bottom of this page. Measurements are available on the following six variables:

`chd`	response, coronary heart disease (1=yes, 0=no)
`famhist`	family history of heart disease, a factor with levels `Absent Present`
`age`	age (years)
`tobacco`	cumulative lifetime tobacco consumption (kg)
`ldl`	low density lipoprotein cholesterol (mmol/L)
`sbp`	systolic blood pressure (mmHg)

(a) For each of the six variables, state whether they are categorical or numerical.

(b) Consider the generalized linear model fitted in the R excerpt below. Write down all its components explicitly in mathematical notation, that is

- the linear predictor,
- the response function,
- the distributional assumption.

(c) Give an interpretation of the estimated parameters for `famhist` and `tobacco`.

(d) Compute the fitted probability of coronary heart disease for the individual in the row labelled “5”.

```
> head(heart)
  chd famhist age tobacco  ldl sbp
1   1 Present  52   12.00 5.73 160
2   1  Absent  63    0.01 4.41 144
3   0 Present  46    0.08 3.48 118
4   1 Present  58    7.50 6.41 170
5   1 Present  49   13.60 3.50 134
6   0 Present  45    6.20 6.47 132
> heart.glm <- glm(chd ~ famhist + age + tobacco + ldl + sbp,
                   data = heart, family = binomial(link=logit))
> heart.glm$coef
  (Intercept) famhistPresent            age        tobacco
 -4.770967133    0.930220852    0.041471717    0.080382564
          ldl            sbp
  0.164818467    0.004976018
```

5. (a) Define the form of probability density/mass function corresponding to members of the exponential dispersion family of distributions. Explain where appropriate any parameters used.

   (b) The Geometric distribution has probability mass function:

$$P(y \,|\, \pi) = (1-\pi)^{y-1}\pi$$

   where $\pi \in [0,1]$ and y is a positive integer. Show that it is a member of the exponential dispersion family, being sure to identify all the constituent parameters defined in part (a).

   (c) Using only properties of the exponential dispersion family, derive the mean and variance of the Geometric distribution in terms of π.

   (d) Hence, what is the natural link function when using a Geometrically distributed response in a generalized linear model?

6. A study at the Hospital del Mar, Barcelona, collected $n = 82$ observations on the length of time that patients remain in hospital. The following 4 variables were collected:

`los`	response, the total number of days the patient spent in hospital
`age`	the age of the patient, subtracted from 55
`sex`	the sex of the patient: a factor with levels 1=male, 2=female
`ward`	the type of ward in the hospital: a factor with levels 1=medical, 2=surgical, 3=others

A generalized linear model with Gamma-distributed response and log-link was fitted, and an analysis of deviance was carried out. Some results are provided in the (edited) R output below, which needs to be used to answer the questions.

(a) Complete the missing values `A`, `B`, `C`, and `D` in the row for `sex`.

(b) Test the smaller model M_0, which only contains an intercept and the covariate `age`, against the full model M_1 fitted below, at the 5% level of significance.

(c) Comment on the value of the estimated dispersion parameter. What does it suggest?

```
> fit <- glm(los ~ age + sex + ward, family = Gamma(link=log))
> summary(fit)$dispersion
[1] 0.9856213
> anova(fit)
Analysis of Deviance Table
Model: Gamma, link: log
Response: los
Terms added sequentially (first to last)


     Df Deviance Resid. Df Resid. Dev
NULL                    81     88.268
age   1  10.2038        80     78.064
sex   A        B         C          D
ward  2   1.0876        77     71.719
```

SECTION B

7. Let $x_1, x_2, ..., x_n$ be independent and identically distributed random variables following a distribution f_θ which is labeled by a d-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^d$, and which admits a PDF $f(\cdot|\theta)$. Let $\tilde{\theta}_n$ be a strongly consistent sequence such that

$$\sqrt{n}(\tilde{\theta}_n - \theta_0) \xrightarrow{D} \mathrm{N}(0, \Sigma(\theta_0))$$

for some $\Sigma(\theta_0) > 0$, where θ_0 is the true value of the parameter θ. Assume that the regularity assumptions of Cramer's theorem giving the asymptotic distribution of a root of the likelihood equation are satisfied. Consider the following one-step estimator based on the Fisher scoring algorithm:

$$\breve{\theta}_n = \tilde{\theta}_n + \frac{1}{n}\mathcal{I}(\tilde{\theta}_n)^{-1}\dot{\ell}_n(\tilde{\theta}_n) \quad ,$$

where $\ell_n(\theta) = \log(\prod_{i=1}^n f(x_i|\theta))$ is the log-likelihood at θ, and $\mathcal{I}(\theta)$ is Fisher's information for one observation. Assume that the likelihood equations have a unique root giving the maximum likelihood estimator (MLE), denoted $\hat{\theta}_n$.

(a) State the conclusion of Cramer's theorem regarding the asymptotic distribution of the MLE, and give an argument for the MLE $\hat{\theta}_n$ being a more desirable estimator than $\tilde{\theta}_n$.

(b) Expand $\dot{\ell}_n(\tilde{\theta}_n)$ around $\hat{\theta}_n$ by using the mean value theorem.

(c) Show that

$$\sqrt{n}(\breve{\theta}_n - \hat{\theta}_n) = \left(I + \mathcal{I}(\tilde{\theta}_n)^{-1} \int_0^1 \frac{1}{n}\ddot{\ell}_n(\hat{\theta}_n + u(\tilde{\theta}_n - \hat{\theta}_n))\mathrm{d}u\right) \times \left(\sqrt{n}(\tilde{\theta}_n - \theta_0) - \sqrt{n}(\hat{\theta}_n - \theta_0)\right)$$

(d) Show that $\breve{\theta}_n$ is an asymptotically efficient estimator.

8. Suppose a certain disease can be characterized as being of high or low severity. The patients have an option to visit one of two hospitals for treatment: a public or private hospital. The outcome of the treatment is binary: success or failure. We have collected a random sample of patients, and we classified them according to the following three variables: the hospital that they visited (X) with levels (private or public), the severity of their disease (Z) with levels (high or low), and the outcome of their treatment (Y) with levels (success or failure). The data-set is presented in Table 2.

hospital visited (X)	treatment outcome (Y)	disease severity (Z)	counts
public	success	low	18
public	failure	low	2
private	success	low	64
private	failure	low	16
public	success	high	32
public	failure	high	48
private	success	high	4
private	failure	high	16

Table 2: Dataset

(a) Calculate the marginal XY-contingency table of the observed counts. Calculate a 95% confidence interval for the marginal odds ratio of the hospital visited and treatment outcome, and based on this, infer whether the hospital visited and treatment outcome are dependent or not. What does the estimated odds ratio of the hospital visited and treatment outcome tell us about the relation between hospital visited and treatment outcome?

(b) We are interested in checking the hypothesis that the hospital visited, treatment outcome, and disease severity are mutually independent against the hypothesis that the hospital visited and the treatment outcome are conditionally independent given the disease severity. State the log-linear model equations for the two associations under consideration. Perform a statistical test at significance level 5% in order to answer the scientific question. Justify the choice of test.

(c) Compute the conditional odds ratio of the hospital visited and treatment outcome at each level of disease severity. What do these conditional odds say about the association between the classification variables involved?

(d) Calculate the marginal odds ratio of the hospital visited and disease severity. What does this odds ratio imply about the association between the hospital visited and the disease severity?

(e) Compare the inferential results from parts (a) and (c). Explain why this behaviour might occur in our data analysis.

9. By the 10th January 2019, each football team in the UK Premier League had played 21 games. A simple model, 'Model 1', is proposed which assigns each team a 'strength', β_j, where we model the probability of a home team win by:

$$p(j,k) := P(\text{home team } j \text{ beats away team } k) = \frac{e^{\beta_0+\beta_j-\beta_k}}{1+e^{\beta_0+\beta_j-\beta_k}} \qquad (1)$$

with β_0 being a fixed constant (intercept). The first column in the table **on the next page** gives the intercept and strengths when the above model was fitted.

(a) How would you construct a design matrix in order to treat this as a standard generalized linear model? Explain how to interpret the intercept in this setup.

(b) The next match was on 12th January 2019, when Newcastle were defeated playing away against Chelsea. Bookies were offering odds on Chelsea winning of 5:1. Does this simple model agree with these odds?

To use the full score we treat a match as grouped data on goals. Let m_i be the number of goals in game i, and let Y_{ir} be a Bernoulli random variable that is 1 if the r^{th} goal of game i is scored by the home team and 0 if scored by the away team.

'Model 2' models the proportion of goals scored by the home team in game i as:

$$\bar{Y}_i = \frac{1}{m_i}\sum_{r=1}^{m_i} Y_{ir} \sim \frac{1}{m_i}\text{Bin}(m_i, p(j,k))$$

where now $p(j,k)$ is the probability that conditional on a goal being scored, it is scored by the home team, j rather than the away team, k. This is modelled in the same way as in equation (1). The goal scoring strengths are shown in the second column of the table **on the next page**, followed by R code used to fit this to `model2`.

(c) Prove that the Binomial distribution rescaled in this way is in the exponential dispersion family. (Hint: you need to show that a Binomial with my 'successes' in m trials can be formulated as an EDF for data y).

(d) On 12th January 2019, the score was (Chelsea, home) 2–1 (Newcastle, away).

i. Using the estimated parameter values and conditional on there being 3 goals, what is the probability of this score?

ii. If the probability of m goals being scored in the game is as in the table below, what are the odds on a Chelsea win?

m	0	1	2	3
$P(m \text{ goals})$	0.05	0.25	0.4	0.3

(e) In this data Liverpool were top of the table. Perform a hypothesis test at the 5% level of significance for whether Chelsea (4^{th} in the league) and Liverpool are actually equally strong (ie $H_0 : \beta_{\text{Chelsea}} = \beta_{\text{Liverpool}}$). Note that the model with the constraint H_0 is the model in `model2e` on the next page.

(f) Find an expression for the deviance components in Model 2 and hence find the deviance residual for the Chelsea–Newcastle game in (d).

Team	β, Model 1	β, Model 2
Liverpool	1.555	1.193
Manchester City	1.009	0.536
Tottenham	0.063	0.376
Arsenal	—	—
Chelsea	−0.498	0.386
Manchester United	−0.711	−0.213
Watford	−0.947	−0.574
Wolves	−1.117	−0.453
Leicester	−1.208	−0.353
Everton	−1.348	−0.522
West Ham	−1.401	−0.693
Newcastle	−1.507	−0.934
Brighton	−1.602	−0.740
Bournemouth	−1.752	−0.680
Cardiff	−1.981	−1.149
Crystal Palace	−2.032	−0.604
Burnley	−2.165	−1.258
Southampton	−2.500	−0.904
Fulham	−2.595	−1.434
Huddersfield	−2.958	−1.509
Intercept, β_0	−0.284	0.167

```
> model2 <- glm(cbind(HomeGoals, AwayGoals) ~ .,
              family = binomial(link=logit), data = football)
> model2$deviance
[1] 189.2149
> model2e <- glm(cbind(HomeGoals, AwayGoals) ~
               . - Liverpool - Chelsea + I(Liverpool+Chelsea),
               family = binomial(link=logit), data = football)
> model2e$deviance
[1] 192.3242
```

10. Consider a generalized linear model with inverse Gaussian distributed response, i.e. with probability density function:

$$P(y \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi y^3}}\, \exp\Big\{-\Big(\frac{\lambda(y-\mu)^2}{2\mu^2 y}\Big)\Big\}$$

where $y \in \mathbb{R}_{\geq 0}$, and $\mu, \lambda \in \mathbb{R}_{>0}$; and $E[y|x, \beta] = h(\beta^T x)$ for some response function h.

(a) The inverse Gaussian distribution forms an exponential dispersion family with dispersion parameter $\phi = 2/\lambda$. Identify the natural parameter θ and the log normalizer $b(\theta)$, and hence derive the canonical link for this model. Why is this choice of link function problematic in general?

From now on, use the log link function.

(b) Given data $\{(x_i, y_i)\}_{i\in[1..n]}$, write down the log likelihood for the model and derive the score function.

(c) Derive the observed Fisher Information $F_{\text{obs}}(\boldsymbol{\beta})$ and the (expected) Fisher Information $F(\boldsymbol{\beta})$.

(d) We are given measurements of peak ground acceleration `a` (in units of g) resulting from 10 different seismic events at different distances `d` (in tens of km) from the observation station:

d	1.20	12.3	1.96	9.10	3.29	0.38	2.20	1.22	2.90	4.92
a	0.359	0.003	0.200	0.039	0.064	0.640	0.150	0.097	0.039	0.017

In order to try to estimate the attenuating effect of distance on acceleration, a generalized linear model of the type described above is fitted to these data, with linear predictor $\eta = \beta_1 + \beta_2$`d`. The estimates for β were found to be $\hat{\beta}_1 = 3.172$ and $\hat{\beta}_2 = -0.729$.

(i) How far must one travel in order to halve the expected peak acceleration?

(ii) Compute the expected Fisher Information matrix.

(iii) Provide an approximate test of $H_0 : \beta_2 = 0$, and comment on the significance level.

(iv) Give the expected value of `a` and an approximate 95% confidence interval for this expected value, when `d` $= 0.5$.

(v) Would you trust this expected value in practice? Why?

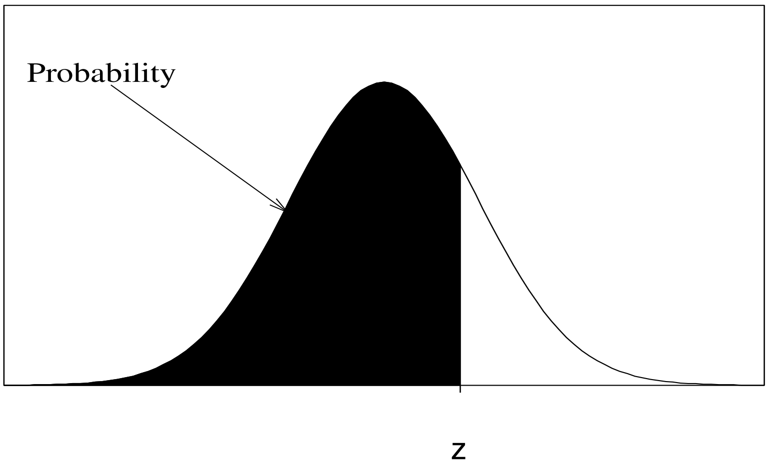
Probabilities for the standard normal distribution

Table entry for z is the probability lying to the left of z, i.e. $\Phi(z)$.

For $z > 3$,

$$1 - \Phi(z) \approx \frac{1}{\sqrt{2\pi}z} e^{-\frac{1}{2}z^2}$$

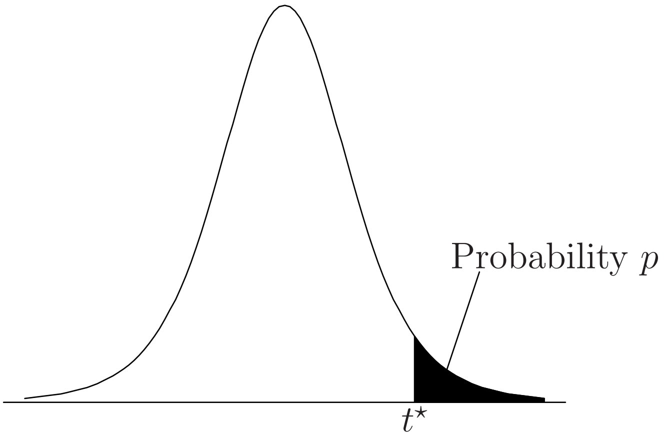
is accurate to within 10% of the true value.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Probabilities for the t-distribution

Table entry for p and C is the point $t^\star$ with probability p lying above it and probability C lying between $-t^\star$ and $t^\star$



	Tail probability p											
df	.25	.2	.15	.1	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.895	31.821	63.657	127.321	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70 %	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

F distribution critical values

Denominator degrees of freedom		Degrees of freedom in the numerator								
	p	1	2	3	4	5	6	7	8	9
2	0.1	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	0.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	0.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	0.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	0.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	0.1	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	0.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	0.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	0.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	0.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	0.1	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	0.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	0.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	0.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	0.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	0.1	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	0.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	0.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	0.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	0.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24
6	0.1	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	0.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	0.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	0.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	0.001	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69
7	0.1	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	0.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	0.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	0.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	0.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33
8	0.1	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
	0.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
	0.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
	0.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
	0.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77
9	0.1	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
	0.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
	0.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
	0.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
	0.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11
10	0.1	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
	0.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
	0.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
	0.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
	0.001	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96
11	0.1	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27
	0.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
	0.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59
	0.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
	0.001	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12
12	0.1	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
	0.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
	0.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
	0.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
	0.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48
13	0.1	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	0.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	0.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
	0.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	0.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98

F distribution critical values

Denominator degrees of freedom		Degrees of freedom in the numerator								
	p	1	2	3	4	5	6	7	8	9
14	0.1	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	0.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	0.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
	0.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	0.1	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	0.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	0.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
	0.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	0.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	0.1	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	0.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	0.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
	0.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	0.1	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	0.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	0.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
	0.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
	0.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75
18	0.1	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	0.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
	0.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
	0.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56
19	0.1	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	0.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
	0.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
	0.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	0.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39
20	0.1	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	0.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
	0.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
	0.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24
21	0.1	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
	0.05	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
	0.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80
	0.01	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
	0.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11
22	0.1	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	0.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	0.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
	0.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	0.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99
23	0.1	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	0.05	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
	0.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73
	0.01	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	0.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89
24	0.1	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	0.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
	0.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
	0.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	0.001	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80
25	0.1	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
	0.05	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
	0.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68
	0.01	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
	0.001	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71

F distribution critical values

		Degrees of freedom in the numerator								
Denominator degrees of freedom	p	1	2	3	4	5	6	7	8	9
26	0.1	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
	0.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
	0.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
	0.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
	0.001	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64
27	0.1	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
	0.05	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
	0.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63
	0.01	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
	0.001	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57
28	0.1	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
	0.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
	0.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
	0.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
	0.001	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50
29	0.1	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
	0.05	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
	0.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59
	0.01	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
	0.001	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45
30	0.1	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
	0.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
	0.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
	0.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
	0.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39
40	0.1	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
	0.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
	0.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
	0.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
	0.001	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02
50	0.1	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76
	0.05	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
	0.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
	0.01	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78
	0.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82
60	0.1	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
	0.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
	0.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
	0.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
	0.001	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69
100	0.1	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69
	0.05	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
	0.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
	0.01	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
	0.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44
200	0.1	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66
	0.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
	0.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
	0.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
	0.001	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26
1000	0.1	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.64
	0.05	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
	0.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13
	0.01	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43
	0.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13

F distribution critical values

		Degrees of freedom in the numerator									
Denominator degrees of freedom	*p*	10	11	12	13	14	15	16	17	18	19
2	0.1	9.39	9.40	9.41	9.41	9.42	9.42	9.43	9.43	9.44	9.44
	0.05	19.40	19.40	19.41	19.42	19.42	19.43	19.43	19.44	19.44	19.44
	0.025	39.40	39.41	39.41	39.42	39.43	39.43	39.44	39.44	39.44	39.45
	0.01	99.40	99.41	99.42	99.42	99.43	99.43	99.44	99.44	99.44	99.45
	0.001	999.40	999.41	999.42	999.42	999.43	999.43	999.44	999.44	999.44	999.45
3	0.1	5.23	5.22	5.22	5.21	5.20	5.20	5.20	5.19	5.19	5.19
	0.05	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67
	0.025	14.42	14.37	14.34	14.30	14.28	14.25	14.23	14.21	14.20	14.18
	0.01	27.23	27.13	27.05	26.98	26.92	26.87	26.83	26.79	26.75	26.72
	0.001	129.25	128.74	128.32	127.96	127.64	127.37	127.14	126.93	126.74	126.57
4	0.1	3.92	3.91	3.90	3.89	3.88	3.87	3.86	3.86	3.85	3.85
	0.05	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81
	0.025	8.84	8.79	8.75	8.71	8.68	8.66	8.63	8.61	8.59	8.58
	0.01	14.55	14.45	14.37	14.31	14.25	14.20	14.15	14.11	14.08	14.05
	0.001	48.05	47.70	47.41	47.16	46.95	46.76	46.60	46.45	46.32	46.21
5	0.1	3.30	3.28	3.27	3.26	3.25	3.24	3.23	3.22	3.22	3.21
	0.05	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57
	0.025	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34
	0.01	10.05	9.96	9.89	9.82	9.77	9.72	9.68	9.64	9.61	9.58
	0.001	26.92	26.65	26.42	26.22	26.06	25.91	25.78	25.67	25.57	25.48
6	0.1	2.94	2.92	2.90	2.89	2.88	2.87	2.86	2.85	2.85	2.84
	0.05	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88
	0.025	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18
	0.01	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45	7.42
	0.001	18.41	18.18	17.99	17.82	17.68	17.56	17.45	17.35	17.27	17.19
7	0.1	2.70	2.68	2.67	2.65	2.64	2.63	2.62	2.61	2.61	2.60
	0.05	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46
	0.025	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48
	0.01	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18
	0.001	14.08	13.88	13.71	13.56	13.43	13.32	13.23	13.14	13.06	12.99
8	0.1	2.54	2.52	2.50	2.49	2.48	2.46	2.45	2.45	2.44	2.43
	0.05	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16
	0.025	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02
	0.01	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38
	0.001	11.54	11.35	11.19	11.06	10.94	10.84	10.75	10.67	10.60	10.54
9	0.1	2.42	2.40	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.30
	0.05	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95
	0.025	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68
	0.01	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83
	0.001	9.89	9.72	9.57	9.44	9.33	9.24	9.15	9.08	9.01	8.95
10	0.1	2.32	2.30	2.28	2.27	2.26	2.24	2.23	2.22	2.22	2.21
	0.05	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79
	0.025	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44
	0.01	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43
	0.001	8.75	8.59	8.45	8.32	8.22	8.13	8.05	7.98	7.91	7.86
11	0.1	2.25	2.23	2.21	2.19	2.18	2.17	2.16	2.15	2.14	2.13
	0.05	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66
	0.025	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24
	0.01	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12
	0.001	7.92	7.76	7.63	7.51	7.41	7.32	7.24	7.17	7.11	7.06
12	0.1	2.19	2.17	2.15	2.13	2.12	2.10	2.09	2.08	2.08	2.07
	0.05	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56
	0.025	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09
	0.01	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88
	0.001	7.29	7.14	7.00	6.89	6.79	6.71	6.63	6.57	6.51	6.45
13	0.1	2.14	2.12	2.10	2.08	2.07	2.05	2.04	2.03	2.02	2.01
	0.05	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47
	0.025	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96
	0.01	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69
	0.001	6.80	6.65	6.52	6.41	6.31	6.23	6.16	6.09	6.03	5.98

F distribution critical values

			Degrees of freedom in the numerator									
		p	10	11	12	13	14	15	16	17	18	19
Denominator degrees of freedom	14	0.1	2.10	2.07	2.05	2.04	2.02	2.01	2.00	1.99	1.98	1.97
		0.05	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40
		0.025	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86
		0.01	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53
		0.001	6.40	6.26	6.13	6.02	5.93	5.85	5.78	5.71	5.66	5.60
	15	0.1	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93
		0.05	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34
		0.025	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77
		0.01	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40
		0.001	6.08	5.94	5.81	5.71	5.62	5.54	5.46	5.40	5.35	5.29
	16	0.1	2.03	2.01	1.99	1.97	1.95	1.94	1.93	1.92	1.91	1.90
		0.05	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29
		0.025	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70
		0.01	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28
		0.001	5.81	5.67	5.55	5.44	5.35	5.27	5.20	5.14	5.09	5.04
	17	0.1	2.00	1.98	1.96	1.94	1.93	1.91	1.90	1.89	1.88	1.87
		0.05	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24
		0.025	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63
		0.01	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19
		0.001	5.58	5.44	5.32	5.22	5.13	5.05	4.99	4.92	4.87	4.82
	18	0.1	1.98	1.95	1.93	1.92	1.90	1.89	1.87	1.86	1.85	1.84
		0.05	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20
		0.025	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58
		0.01	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10
		0.001	5.39	5.25	5.13	5.03	4.94	4.87	4.80	4.74	4.68	4.63
	19	0.1	1.96	1.93	1.91	1.89	1.88	1.86	1.85	1.84	1.83	1.82
		0.05	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17
		0.025	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53
		0.01	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03
		0.001	5.22	5.08	4.97	4.87	4.78	4.70	4.64	4.58	4.52	4.47
	20	0.1	1.94	1.91	1.89	1.87	1.86	1.84	1.83	1.82	1.81	1.80
		0.05	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14
		0.025	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48
		0.01	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96
		0.001	5.08	4.94	4.82	4.72	4.64	4.56	4.49	4.44	4.38	4.33
	21	0.1	1.92	1.90	1.87	1.86	1.84	1.83	1.81	1.80	1.79	1.78
		0.05	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11
		0.025	2.73	2.68	2.64	2.60	2.56	2.53	2.51	2.48	2.46	2.44
		0.01	3.31	3.24	3.17	3.12	3.07	3.03	2.99	2.96	2.93	2.90
		0.001	4.95	4.81	4.70	4.60	4.51	4.44	4.37	4.31	4.26	4.21
	22	0.1	1.90	1.88	1.86	1.84	1.83	1.81	1.80	1.79	1.78	1.77
		0.05	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08
		0.025	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41
		0.01	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85
		0.001	4.83	4.70	4.58	4.49	4.40	4.33	4.26	4.20	4.15	4.10
	23	0.1	1.89	1.87	1.84	1.83	1.81	1.80	1.78	1.77	1.76	1.75
		0.05	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06
		0.025	2.67	2.62	2.57	2.53	2.50	2.47	2.44	2.42	2.39	2.37
		0.01	3.21	3.14	3.07	3.02	2.97	2.93	2.89	2.86	2.83	2.80
		0.001	4.73	4.60	4.48	4.39	4.30	4.23	4.16	4.10	4.05	4.00
	24	0.1	1.88	1.85	1.83	1.81	1.80	1.78	1.77	1.76	1.75	1.74
		0.05	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04
		0.025	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35
		0.01	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76
		0.001	4.64	4.51	4.39	4.30	4.21	4.14	4.07	4.02	3.96	3.92
	25	0.1	1.87	1.84	1.82	1.80	1.79	1.77	1.76	1.75	1.74	1.73
		0.05	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02
		0.025	2.61	2.56	2.51	2.48	2.44	2.41	2.38	2.36	2.34	2.32
		0.01	3.13	3.06	2.99	2.94	2.89	2.85	2.81	2.78	2.75	2.72
		0.001	4.56	4.42	4.31	4.22	4.13	4.06	3.99	3.94	3.88	3.84

F distribution critical values

Denominator degrees of freedom		Degrees of freedom in the numerator									
	p	10	11	12	13	14	15	16	17	18	19
	0.1	1.86	1.83	1.81	1.79	1.77	1.76	1.75	1.73	1.72	1.71
	0.05	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00
26	0.025	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29
	0.01	3.09	3.02	2.96	2.90	2.86	2.81	2.78	2.75	2.72	2.69
	0.001	4.48	4.35	4.24	4.14	4.06	3.99	3.92	3.86	3.81	3.77
	0.1	1.85	1.82	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70
	0.05	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99
27	0.025	2.57	2.51	2.47	2.43	2.39	2.36	2.34	2.31	2.29	2.27
	0.01	3.06	2.99	2.93	2.87	2.82	2.78	2.75	2.71	2.68	2.66
	0.001	4.41	4.28	4.17	4.08	3.99	3.92	3.86	3.80	3.75	3.70
	0.1	1.84	1.81	1.79	1.77	1.75	1.74	1.73	1.71	1.70	1.69
	0.05	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97
28	0.025	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25
	0.01	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63
	0.001	4.35	4.22	4.11	4.01	3.93	3.86	3.80	3.74	3.69	3.64
	0.1	1.83	1.80	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.68
	0.05	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96
29	0.025	2.53	2.48	2.43	2.39	2.36	2.32	2.30	2.27	2.25	2.23
	0.01	3.00	2.93	2.87	2.81	2.77	2.73	2.69	2.66	2.63	2.60
	0.001	4.29	4.16	4.05	3.96	3.88	3.80	3.74	3.68	3.63	3.59
	0.1	1.82	1.79	1.77	1.75	1.74	1.72	1.71	1.70	1.69	1.68
	0.05	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95
30	0.025	2.51	2.46	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21
	0.01	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57
	0.001	4.24	4.11	4.00	3.91	3.82	3.75	3.69	3.63	3.58	3.53
	0.1	1.76	1.74	1.71	1.70	1.68	1.66	1.65	1.64	1.62	1.61
	0.05	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87	1.85
40	0.025	2.39	2.33	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09
	0.01	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42	2.39
	0.001	3.87	3.75	3.64	3.55	3.47	3.40	3.34	3.28	3.23	3.19
	0.1	1.73	1.70	1.68	1.66	1.64	1.63	1.61	1.60	1.59	1.58
	0.05	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81	1.80
50	0.025	2.32	2.26	2.22	2.18	2.14	2.11	2.08	2.06	2.03	2.01
	0.01	2.70	2.63	2.56	2.51	2.46	2.42	2.38	2.35	2.32	2.29
	0.001	3.67	3.55	3.44	3.35	3.27	3.20	3.14	3.09	3.04	2.99
	0.1	1.71	1.68	1.66	1.64	1.62	1.60	1.59	1.58	1.56	1.55
	0.05	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78	1.76
60	0.025	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96
	0.01	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25	2.22
	0.001	3.54	3.42	3.32	3.23	3.15	3.08	3.02	2.96	2.91	2.87
	0.1	1.66	1.64	1.61	1.59	1.57	1.56	1.54	1.53	1.52	1.50
	0.05	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73	1.71	1.69
100	0.025	2.18	2.12	2.08	2.04	2.00	1.97	1.94	1.91	1.89	1.87
	0.01	2.50	2.43	2.37	2.31	2.27	2.22	2.19	2.15	2.12	2.09
	0.001	3.30	3.18	3.07	2.99	2.91	2.84	2.78	2.73	2.68	2.63
	0.1	1.63	1.60	1.58	1.56	1.54	1.52	1.51	1.49	1.48	1.47
	0.05	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67	1.66	1.64
200	0.025	2.11	2.06	2.01	1.97	1.93	1.90	1.87	1.84	1.82	1.80
	0.01	2.41	2.34	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.00
	0.001	3.12	3.00	2.90	2.82	2.74	2.67	2.61	2.56	2.51	2.46
	0.1	1.61	1.58	1.55	1.53	1.51	1.49	1.48	1.46	1.45	1.44
	0.05	1.84	1.80	1.76	1.73	1.70	1.68	1.65	1.63	1.61	1.60
1000	0.025	2.06	2.01	1.96	1.92	1.88	1.85	1.82	1.79	1.77	1.74
	0.01	2.34	2.27	2.20	2.15	2.10	2.06	2.02	1.98	1.95	1.92
	0.001	2.99	2.87	2.77	2.69	2.61	2.54	2.48	2.43	2.38	2.34

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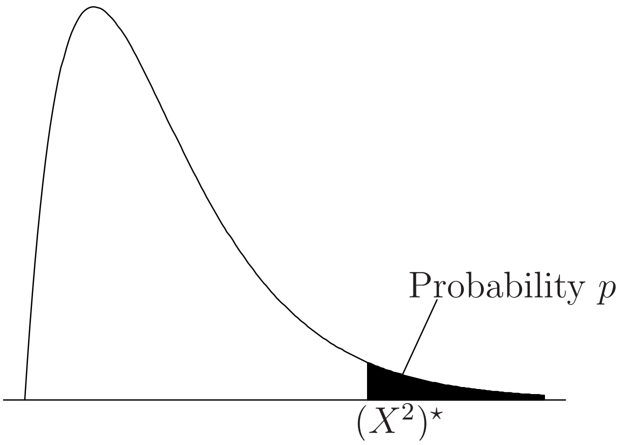
R Development Core Team (2009). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org.

Probabilities for the χ^2-distribution

Table entry for p is the point $(X^2)^\star$ with probability p lying above it



	Tail probability p											
df	.995	.975	.25	.2	.1	.05	.025	.01	.005	.0025	.001	.0005
1	0.000039	0.00098	1.32	1.64	2.71	3.84	5.02	6.63	7.88	9.14	10.83	12.12
2	0.010	0.051	2.77	3.22	4.61	5.99	7.38	9.21	10.60	11.98	13.82	15.20
3	0.072	0.22	4.11	4.64	6.25	7.81	9.35	11.34	12.84	14.32	16.27	17.73
4	0.21	0.48	5.39	5.99	7.78	9.49	11.14	13.28	14.86	16.42	18.47	20.00
5	0.41	0.83	6.63	7.29	9.24	11.07	12.83	15.09	16.75	18.39	20.52	22.11
6	0.68	1.24	7.84	8.56	10.64	12.59	14.45	16.81	18.55	20.25	22.46	24.10
7	0.99	1.69	9.04	9.80	12.02	14.07	16.01	18.48	20.28	22.04	24.32	26.02
8	1.34	2.18	10.22	11.03	13.36	15.51	17.53	20.09	21.95	23.77	26.12	27.87
9	1.73	2.70	11.39	12.24	14.68	16.92	19.02	21.67	23.59	25.46	27.88	29.67
10	2.16	3.25	12.55	13.44	15.99	18.31	20.48	23.21	25.19	27.11	29.59	31.42
11	2.60	3.82	13.70	14.63	17.28	19.68	21.92	24.72	26.76	28.73	31.26	33.14
12	3.07	4.40	14.85	15.81	18.55	21.03	23.34	26.22	28.30	30.32	32.91	34.82
13	3.57	5.01	15.98	16.98	19.81	22.36	24.74	27.69	29.82	31.88	34.53	36.48
14	4.07	5.63	17.12	18.15	21.06	23.68	26.12	29.14	31.32	33.43	36.12	38.11
15	4.60	6.26	18.25	19.31	22.31	25.00	27.49	30.58	32.80	34.95	37.70	39.72
16	5.14	6.91	19.37	20.47	23.54	26.30	28.85	32.00	34.27	36.46	39.25	41.31
17	5.70	7.56	20.49	21.61	24.77	27.59	30.19	33.41	35.72	37.95	40.79	42.88
18	6.26	8.23	21.60	22.76	25.99	28.87	31.53	34.81	37.16	39.42	42.31	44.43
19	6.84	8.91	22.72	23.90	27.20	30.14	32.85	36.19	38.58	40.88	43.82	45.97
20	7.43	9.59	23.83	25.04	28.41	31.41	34.17	37.57	40.00	42.34	45.31	47.50
21	8.03	10.28	24.93	26.17	29.62	32.67	35.48	38.93	41.40	43.78	46.80	49.01
22	8.64	10.98	26.04	27.30	30.81	33.92	36.78	40.29	42.80	45.20	48.27	50.51
23	9.26	11.69	27.14	28.43	32.01	35.17	38.08	41.64	44.18	46.62	49.73	52.00
24	9.89	12.40	28.24	29.55	33.20	36.42	39.36	42.98	45.56	48.03	51.18	53.48
25	10.52	13.12	29.34	30.68	34.38	37.65	40.65	44.31	46.93	49.44	52.62	54.95
26	11.16	13.84	30.43	31.79	35.56	38.89	41.92	45.64	48.29	50.83	54.05	56.41
27	11.81	14.57	31.53	32.91	36.74	40.11	43.19	46.96	49.64	52.22	55.48	57.86
28	12.46	15.31	32.62	34.03	37.92	41.34	44.46	48.28	50.99	53.59	56.89	59.30
29	13.12	16.05	33.71	35.14	39.09	42.56	45.72	49.59	52.34	54.97	58.30	60.73
30	13.79	16.79	34.80	36.25	40.26	43.77	46.98	50.89	53.67	56.33	59.70	62.16
40	20.71	24.43	45.62	47.27	51.81	55.76	59.34	63.69	66.77	69.70	73.40	76.09
50	27.99	32.36	56.33	58.16	63.17	67.50	71.42	76.15	79.49	82.66	86.66	89.56
60	35.53	40.48	66.98	68.97	74.40	79.08	83.30	88.38	91.95	95.34	99.61	102.69
80	51.17	57.15	88.13	90.41	96.58	101.88	106.63	112.33	116.32	120.10	124.84	128.26
100	67.33	74.22	109.14	111.67	118.50	124.34	129.56	135.81	140.17	144.29	149.45	153.17

Mantel-Haenszel test statistic

The statistic is

$$T_{MH} = \frac{[\,\sum_k (n_{11k} - \mu_{11k})\,]^2}{\sum_k \sigma_{11k}^2}$$

where

$$\mu_{11k} = \frac{n_{1+k} n_{+1k}}{n_{++k}}$$

and

$$\sigma_{11k}^2 = \frac{n_{1+k} n_{2+k} n_{+1k} n_{+2k}}{n_{++k}^2 (n_{++k} - 1)}$$