EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4041-WE01

Title: Partial Differential Equations IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Consider the conservation law

$$\begin{cases} \partial_t u + u\partial_x u = 0, \ (x,t) \in \mathbb{R} \times [0,T), \\ u(x,0) = \arctan(x), \ x \in \mathbb{R}. \end{cases} \tag{1}$$

   (a) Find the largest value of $T \geq 0$ for which the system (1) has a classical solution $u : \mathbb{R} \times [0,T) \to \mathbb{R}$;

   (b) Give a sketch of characteristics for problem (1);

   (c) Find an explicit equation for the function u which does not contain partial derivatives.

2. Consider the conservation law

$$\begin{cases} \partial_t u - \sin u \partial_x u = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = x + \frac{\pi}{2}, \ x \in \mathbb{R}. \end{cases} \tag{2}$$

   (a) Find the characteristics and sketch the solution $u(x,t)$ at a time moment $t > 0$.

   (b) Based on the sketch conclude whether we may expect the existence of a classical solution to problem (2) for all $t > 0$.

3. Consider the 1st order scalar quasi-linear PDE

$$\begin{cases} -x\partial_x u + 2y\partial_y u = -x^2 - y^2, \ (x,y) \in \mathbb{R} \times \{y : \ y > 1\}, \\ u(x,1) = x^2, \ x \in \mathbb{R}. \end{cases} \tag{3}$$

   (a) Write down the system of characteristic ODEs, including initial data, corresponding to the problem (3);

   (b) Find the solution $u(x,y)$.

4. Consider Poisson's equation with Neumann boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary, $n \geq 2$, and $\boldsymbol{n}$ is the outward-pointing unit normal vector field to $\partial\Omega$. The given data for the problem are $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, and the unknown is $u : \overline{\Omega} \to \mathbb{R}$.

   (a) Prove that a necessary condition for the existence of a solution to (4) is

$$\int_\Omega f \, d\boldsymbol{x} + \int_{\partial\Omega} g \, dS = 0.$$

   (b) Show that if we find one solution of (4) then we can derive infinitely many solutions.

5. (a) If u is harmonic in $|x| < 1$, $|y| < 1$, and $u = x^2 + y^2$ on the boundary lines $|x| = 1$ and $|y| = 1$, find lower and upper bounds for $u(0, 0)$.

   (b) Verify that

$$v = \frac{47}{40} - \frac{1}{5}(x^4 - 6x^2y^2 + y^4)$$

   is harmonic and that $-0.025 \leq v - 1 - x^2 \leq 0.025$ when $|x| < 1$ and $|y| = 1$.

6. Let Φ be the fundamental solution of Poisson's equation in $\mathbb{R}^3$:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi} \frac{1}{|\boldsymbol{x}|}.$$

   (a) Let $R > 0$. Compute $\|\Phi\|_{L^1(B_R(\mathbf{0}))}$.

   (b) Prove that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^3)$.

SECTION B

7. (a) Give the definition of a weak solution to the problem

$$\begin{cases} \partial_t u + \partial_x u^5 = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{5}$$

where $u_0(x) \in L^\infty(\mathbb{R})$.

(b) Find a weak entropy solution to the problem (5), if

$$u_0(x) = \begin{cases} -1, \ x < 0, \\ 0, \ x > 0. \end{cases}$$

(c) Find a weak entropy solution to the problem (5), if

$$u_0(x) = \begin{cases} 0, \ x < 0, \\ -1, \ x > 0. \end{cases}$$

8. (a) Let $\{f_n(x)\}_{n=1}^{\infty}$, $f(x) \in \mathcal{D}'(\mathbb{R})$. Explain what the following convergence means

$$f_n(x) \to f(x) \text{ as } n \to +\infty \text{ in } \mathcal{D}'(\mathbb{R});$$

(b) Let

$$f_n(x) = \begin{cases} n, \ x \in [0, \frac{1}{n}], \\ 0, \ x \in \mathbb{R} \setminus [0, \frac{1}{n}]. \end{cases}$$

Prove that for any $\phi \in \mathcal{D}(\mathbb{R})$, such that $\phi(0) \neq 0$, we have $((f_n)^2, \phi) \to \infty$ as $n \to +\infty$;

(c) Does there exist a limit of $(f_n(x))^2$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$? Justify your answer. Hint: use (b).

(d) Prove that for any $\phi \in \mathcal{D}(\mathbb{R})$

$$((f_n)^2 - n\delta, \phi) \to \phi'(0), \text{ as } n \to +\infty,$$

where $\delta(x)$ is the Dirac delta function.

9. Let $u : \mathbb{R}^n \to [0, \infty)$ be harmonic and non-negative.

(a) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $R > r > 0$, and $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$. Use a mean-value formula to prove that

$$u(\boldsymbol{x}) \leq \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|}\, u(\boldsymbol{y}).$$

(b) Set $r = R - |\boldsymbol{x} - \boldsymbol{y}|$. Show that $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$ and compute

$$\lim_{R\to\infty} \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|}.$$

(c) Conclude that u is constant.

10. Assume that $u \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\hat{u} \in L^1(\mathbb{R})$, and $u' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that

$$\|u\|_{H^1(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + \|u'\|^2_{L^2(\mathbb{R})}\right)^{1/2}.$$

(a) Prove that

$$\|u\|^2_{H^1(\mathbb{R})} = \int_{-\infty}^{\infty} (1 + |\xi|^2)|\hat{u}(\xi)|^2\, d\xi.$$

You can use without proof the fact that the Fourier transform preserves the L^2–norm: $\|\hat{v}\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})}$ for all $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(b) Prove that there exists a constant $C > 0$ such that

$$\|\hat{u}\|_{L^1(\mathbb{R})} \leq C\|u\|_{H^1(\mathbb{R})}.$$

(c) Use the Fourier Inversion Theorem to prove that

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}}\|u\|_{H^1(\mathbb{R})}.$$

SECTION C

11. (a) Compute $(\text{sign}(\sin(x)))'$, where the derivative is understood in the sense of distributions. Here

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

(b) Show that

$$|\sin(x)|'' + |\sin(x)| = 2 \sum_{k=-\infty}^{\infty} \delta(x - k\pi),$$

where derivatives are understood in the sense of distributions.
Hint: first use chain rule to compute $|\sin(x)|'$.

(c) Let $u(x,t) = \delta(x - at)$, where $a \in \mathbb{R}$ is a constant. Show, using the definition of the distributional derivative, that for any $\phi(x,t) \in \mathcal{D}(\mathbb{R} \times (0,T))$ $(T > 0)$ we have

$$\int_0^T (\partial_t u, \phi)\, dt + a \int_0^T (\partial_x u, \phi)\, dt = 0,$$

that is $\partial_t u + a\partial_x u \equiv 0$ on $\mathbb{R} \times (0,T)$ in the sense of distributions.