

## **EXAMINATION PAPER**

Examination Session: May

2019

Year:

Exam Code:

MATH4041-WE01

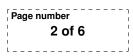
### Title:

# Partial Differential Equations IV

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best <b>TWO</b> answers from Section the best <b>THREE</b> answers from Section <b>AND</b> the answer to the question in Section B and C carry <b>T</b> those in Section A.	on B, ection C.	any marks as
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Revision:



#### SECTION A

1. Consider the conservation law

$$\begin{cases} \partial_t u + u \partial_x u = 0, \ (x, t) \in \mathbb{R} \times [0, T), \\ u(x, 0) = \arctan(x), \ x \in \mathbb{R}. \end{cases}$$
(1)

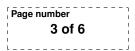
- (a) Find the largest value of  $T \ge 0$  for which the system (1) has a classical solution  $u : \mathbb{R} \times [0, T) \to \mathbb{R};$
- (b) Give a sketch of characteristics for problem (1);
- (c) Find an explicit equation for the function u which does not contain partial derivatives.
- 2. Consider the conservation law

$$\begin{cases} \partial_t u - \sin u \partial_x u = 0, \ (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x, 0) = x + \frac{\pi}{2}, \ x \in \mathbb{R}. \end{cases}$$
(2)

- (a) Find the characteristics and sketch the solution u(x, t) at a time moment t > 0.
- (b) Based on the sketch conclude whether we may expect the existence of a classical solution to problem (2) for all t > 0.
- 3. Consider the 1st order scalar quasi-linear PDE

$$\begin{cases} -x\partial_x u + 2y\partial_y u = -x^2 - y^2, \ (x, y) \in \mathbb{R} \times \{y : \ y > 1\}, \\ u(x, 1) = x^2, \ x \in \mathbb{R}. \end{cases}$$
(3)

- (a) Write down the system of characteristic ODEs, including initial data, corresponding to the problem (3);
- (b) Find the solution u(x, y).





4. Consider Poisson's equation with Neumann boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega,$$
  

$$\nabla u \cdot \boldsymbol{n} = g \quad \text{on } \partial\Omega,$$
(4)

where  $\Omega \subset \mathbb{R}^n$  is an open and bounded set with smooth boundary,  $n \geq 2$ , and  $\boldsymbol{n}$  is the outward-pointing unit normal vector field to  $\partial\Omega$ . The given data for the problem are  $f: \Omega \to \mathbb{R}$  and  $g: \partial\Omega \to \mathbb{R}$ , and the unknown is  $u: \overline{\Omega} \to \mathbb{R}$ .

(a) Prove that a necessary condition for the existence of a solution to (4) is

$$\int_{\Omega} f \, d\boldsymbol{x} + \int_{\partial \Omega} g \, dS = 0.$$

- (b) Show that if we find one solution of (4) then we can derive infinitely many solutions.
- 5. (a) If u is harmonic in |x| < 1, |y| < 1, and  $u = x^2 + y^2$  on the boundary lines |x| = 1 and |y| = 1, find lower and upper bounds for u(0,0).
  - (b) Verify that

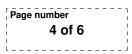
$$v = \frac{47}{40} - \frac{1}{5}(x^4 - 6x^2y^2 + y^4)$$

is harmonic and that  $-0.025 \le v - 1 - x^2 \le 0.025$  when |x| < 1 and |y| = 1.

6. Let  $\Phi$  be the fundamental solution of Poisson's equation in  $\mathbb{R}^3$ :

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi} \frac{1}{|\boldsymbol{x}|}.$$

- (a) Let R > 0. Compute  $\|\Phi\|_{L^1(B_R(\mathbf{0}))}$ .
- (b) Prove that  $\Phi \in L^1_{\text{loc}}(\mathbb{R}^3)$ .



#### SECTION B

7. (a) Give the definition of a weak solution to the problem

$$\begin{cases} \partial_t u + \partial_x u^5 = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases}$$
(5)

where  $u_0(x) \in L^{\infty}(\mathbb{R})$ .

(b) Find a weak entropy solution to the problem (5), if

$$u_0(x) = \begin{cases} -1, \ x < 0, \\ 0, \ x > 0. \end{cases}$$

(c) Find a weak entropy solution to the problem (5), if

$$u_0(x) = \begin{cases} 0, \ x < 0, \\ -1, \ x > 0. \end{cases}$$

8. (a) Let  $\{f_n(x)\}_{n=1}^{\infty}, f(x) \in \mathcal{D}'(\mathbb{R})$ . Explain what the following convergence means

$$f_n(x) \to f(x) \text{ as } n \to +\infty \text{ in } \mathcal{D}'(\mathbb{R});$$

(b) Let

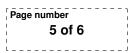
$$f_n(x) = \begin{cases} n, \ x \in [0, \frac{1}{n}], \\ 0, \ x \in \mathbb{R} \setminus [0, \frac{1}{n}]. \end{cases}$$

Prove that for any  $\phi \in \mathcal{D}(\mathbb{R})$ , such that  $\phi(0) \neq 0$ , we have  $((f_n)^2, \phi) \to \infty$  as  $n \to +\infty$ ;

- (c) Does there exist a limit of  $(f_n(x))^2$  in  $\mathcal{D}'(\mathbb{R})$  as  $n \to +\infty$ ? Justify your answer. Hint: use (b).
- (d) Prove that for any  $\phi \in \mathcal{D}(\mathbb{R})$

$$((f_n)^2 - n\delta, \phi) \to \phi'(0), \text{ as } n \to +\infty,$$

where  $\delta(x)$  is the Dirac delta function.





- 9. Let  $u: \mathbb{R}^n \to [0, \infty)$  be harmonic and non-negative.
  - (a) Let  $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ , R > r > 0, and  $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$ . Use a mean-value formula to prove that

$$u(\boldsymbol{x}) \leq \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} u(\boldsymbol{y}).$$

(b) Set  $r = R - |\boldsymbol{x} - \boldsymbol{y}|$ . Show that  $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$  and compute

$$\lim_{R o\infty}rac{|B_R(oldsymbol{y})|}{|B_r(oldsymbol{x})|}.$$

- (c) Conclude that u is constant.
- 10. Assume that  $u \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ ,  $\hat{u} \in L^1(\mathbb{R})$ , and  $u' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Recall that

$$||u||_{H^1(\mathbb{R})} = \left( ||u||^2_{L^2(\mathbb{R})} + ||u'||^2_{L^2(\mathbb{R})} \right)^{1/2}.$$

(a) Prove that

$$||u||_{H^1(\mathbb{R})}^2 = \int_{-\infty}^{\infty} (1+|\xi|^2) |\hat{u}(\xi)|^2 d\xi.$$

You can use without proof the fact that the Fourier transform preserves the  $L^2$ -norm:  $\|\hat{v}\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})}$  for all  $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ .

(b) Prove that there exists a constant C > 0 such that

$$\|\hat{u}\|_{L^1(\mathbb{R})} \le C \|u\|_{H^1(\mathbb{R})}.$$

(c) Use the Fourier Inversion Theorem to prove that

$$\|u\|_{L^{\infty}(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^1(\mathbb{R})}.$$

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#### SECTION C

11. (a) Compute (sign(sin(x)))', where the derivative is understood in the sense of distributions. Here

$$\operatorname{sign}(x) = \begin{cases} 1, \ x > 0, \\ -1, \ x < 0. \end{cases}$$

(b) Show that

$$|\sin(x)|'' + |\sin(x)| = 2\sum_{k=-\infty}^{\infty} \delta(x - k\pi),$$

where derivatives are understood in the sense of distributions. Hint: first use chain rule to compute  $|\sin(x)|'$ .

(c) Let  $u(x,t) = \delta(x-at)$ , where  $a \in \mathbb{R}$  is a constant. Show, using the definition of the distributional derivative, that for any  $\phi(x,t) \in \mathcal{D}(\mathbb{R} \times (0,T))$  (T > 0) we have

$$\int_0^T (\partial_t u, \phi) \, dt + a \int_0^T (\partial_x u, \phi) \, dt = 0,$$

that is  $\partial_t u + a \partial_x u \equiv 0$  on  $\mathbb{R} \times (0, T)$  in the sense of distributions.