Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4051-WE01

Title: General Relativity IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. A two-dimensional space-time has coordinates $x^\mu = (t, x)$ and metric

$$ds^2 = \mathrm{e}^{2t}(dt^2 - dx^2).$$

Different coordinates $\tilde{x}^\mu = (p, q)$ are defined by $p = \exp(t + x)$ and $q = \exp(t - x)$.

   (a) If a vector field V has components $V^\mu = (1, 1)$ with respect to the original coordinates, what are its components $\widetilde{V}^\mu$ with respect to the new coordinates $\tilde{x}^\mu$?

   (b) Compute the components $\tilde{g}_{\mu\nu}$ of the metric with respect to the new coordinates.

2. A four-dimensional space-time is equipped with the metric

$$ds^2 = 2\,dt\,dx - x^2(dx^2 + dy^2 + dz^2) \text{ for } x > 0.$$

   (a) Given a covector field $W_\mu = (1, 1, 1, 1)$, compute the scalar field $f = W_\mu W^\mu$.

   (b) Compute the proper length D of the curve defined by $t(s) = s^3$, $x(s) = y(s) = s$, $z(s) = 0$ for $1 \leq s \leq 2$.

3. (a) Give the definitions of the Lie derivative $\mathcal{L}_V f$ of a scalar field f, and the Lie derivative $(\mathcal{L}_V U)^\mu$ of a vector field U^μ.

   (b) Given that $\mathcal{L}_V$ satisfies the Leibniz rule, derive an expression for the Lie derivative $(\mathcal{L}_V W)_\mu$ of a covector field W_μ.

4. In this problem we work in two-dimensional Minkowski space-time, in Cartesian coordinates, with metric $ds^2 = dt^2 - dx^2$. Consider two twins, Twin A and Twin B. At $t = 0$, both twins are the same age. They now follow two different trajectories through space-time. Twin A follows $t(s_A) = s_A$, $x(s_A) = 0$, while Twin B follows $t(s_B) = s_B$, $x(s_B) = \sin(s_B)$, where s_A and s_B are parameters along the two paths.

   (a) At what coordinate time t_1 do the two twins meet again for the first time?

   (b) Compute the proper time elapsed along each of the two trajectories from $t = 0$ to $t = t_1$.

   (c) Which of the two trajectories that twins A and B follow is a geodesic? When the two twins meet again at $t = t_1$, which twin is older?

5. (a) State the condition for a covector field v_μ to be a Killing covector field.
   (b) Consider flat two-dimensional space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2$. How many (linearly independent) Killing covector fields does this space have?
   (c) Write down these Killing covector fields in polar coordinates.

6. Consider the space-time with metric

$$ds^2 = \frac{1}{\cos^2(\sigma)} \left(dt^2 - d\sigma^2 \right) ,$$

where the spatial coordinate is $\sigma \in (-\pi/2, \pi/2)$.

   (a) Show that the proper distance between the two ends of the space-time at $\sigma = -\pi/2$ and $\sigma = \pi/2$ is infinite, but that a light ray can travel from one end to the other in finite coordinate time t.
   (b) A stationary observer at $\sigma = \sigma_0$ emits a photon of frequency ω_0, which is then observed by a stationary observer at $\sigma = \sigma_1$. Derive a formula for the frequency ω_1 observed at $\sigma = \sigma_1$.

SECTION B

7. A two-dimensional space has coordinates (r, θ) where $r > 0$, and θ is periodic with period 2π. Its metric is $ds^2 = dr^2 + u(r)\, d\theta^2$, where $u(r)$ is a positive function.

   (a) Compute the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$.
   (b) Consider the parallel propagation of a vector V^μ around a closed loop $r = r_0$ constant. We want V^μ to return to its original value after travelling from $\theta = 0$ to $\theta = 2\pi$. What is the most general function $u(r)$ for which this holds, over *all* loops $r =$ constant, subject only to the condition that $u(0) = 0$?
   (c) Find the geodesics of the metric $ds^2 = dr^2 + r\, d\theta^2$ which are given by even functions $r = r(\theta)$. [You might find the following integral useful: $\int dx/\sqrt{x^2 - a^2} = \cosh^{-1}(x/a)$.]

8. (a) The antisymmetric tensor field $F_{\mu\nu}$ satisfies Maxwell's equations $\nabla^\mu F_{\mu\nu} = 0$ and $\nabla_{[\alpha} F_{\mu\nu]} = 0$. Show that $T_{\mu\nu} = F_\mu{}^\alpha F_{\nu\alpha} + m F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}$ satisfies $\nabla^\mu T_{\mu\nu} = 0$, for some constant m which you should determine.

(b) A two-dimensional space with coordinates $(x^1, x^2) = (p, \theta)$ has the metric $ds^2 = dp^2 + \sinh^2(p)\, d\theta^2$ with $p > 0$. Show that $V^\mu = (\cos\theta, k\coth(p)\sin\theta)$ is a Killing vector field, for some value of the constant k which you should determine. You may use the fact that if $f = V_\mu \dot{x}^\mu$ is constant along arbitrary affinely-parametrized geodesics $x^\mu = x^\mu(s)$, then V_μ satisfies the Killing equation (here $\dot{x}^\mu = dx^\mu/ds$ as usual).

(c) Given that the tensor field $P_{\mu\nu}$ satisfies $\nabla^\mu P_{\mu\nu} = 0$, find constants A and B such that

$$\nabla^\mu \nabla_\nu P_{\mu\alpha} = A R_{\nu\beta} P^\beta{}_\alpha + B R_{\mu\nu\alpha\beta} P^{\mu\beta}.$$

9. Consider the space-time with metric

$$ds^2 = (1 + Br^2)^2 (dt^2 - dr^2 - dz^2) - \frac{r^2}{(1 + Br^2)^2}\, d\phi^2,$$

where $B > 0$ is a constant. As usual ϕ is a periodic coordinate with period 2π, and $r \geq 0$ is a radial coordinate.

(a) Identify the coordinate and curvature singularities of this space-time, if any.

(b) The space-time has Killing vector fields associated with translations in each of the t, ϕ, and z directions: write down these Killing vector fields.

(c) Use the Killing vectors to write down the massive geodesic equation in the form

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 + V(r) = 0,$$

where s is an affine parameter along the trajectory, and $V(r)$ can be written in terms of the constants of the motion.

(d) Is it possible for a particle obeying the geodesic equation to move in only the z (and t) directions? If so, find the radial position of the particle $r = r^*$ in terms of the parameters appearing in $V(r)$.

(e) Consider a particle obeying the geodesic equation and moving only in r (and t), with no motion in the ϕ or z directions. If it is displaced slightly away from $r = 0$, it will exhibit oscillations around $r = 0$. Assuming these oscillations to have a small amplitude, compute their frequency (as measured with respect to s).

10. Here we will study cosmology in d space-time dimensions, with coordinates $x^\mu = (t, x^i)$, $i \in \{1, 2, \cdots d-1\}$. Consider the analogue of the FRW metric in d space-time dimensions with no spatial curvature, namely

$$ds^2 = dt^2 - a(t)^2 \left(dx_1^2 + dx_2^2 + \cdots + dx_{d-1}^2\right)$$

The components of the Ricci tensor for this metric are

$$R_{tt} = -(d-1)\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(\ddot{a}a + (d-2)\dot{a}^2\right)\delta_{ij},$$

with all other components zero. As usual, we will assume the matter stress tensor is given by

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - p g^{\mu\nu}, \qquad U^\mu = \delta^\mu_t,$$

where ρ and p are the energy-density and pressure. They are related by an equation of state $p = w\rho$, with w a dimensionless constant.

(a) From the usual Einstein equations, show that the scale factor satisfies the d-dimensional analogue of the FRW equations, namely

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi G}{(d-2)(d-1)}\rho, \qquad \frac{\ddot{a}}{a} = -\frac{8\pi G}{d-2}\left(p + \frac{d-3}{d-1}\rho\right).$$

(b) From these equations, obtain an expression for $\dot{\rho}$ of the form

$$\dot{\rho} + \alpha\frac{\dot{a}}{a}(\rho + p) = 0,$$

where α is a constant that depends on d in a manner that you should determine.

(c) For arbitrary d, determine the form of $\rho(a)$ for matter domination ($w = 0$) and radiation domination ($w = 1/(d-1)$).

(d) Solve the FRW equations to find the time-dependence of a, for arbitrary d, for each of the two cases of matter and radiation domination.