Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4061-WE01

Title:
Advanced Quantum Theory IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. The action for two real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \bigg\{ -\frac{1}{2}m^2\varphi_1(x)^2 - \frac{1}{2}m^2\varphi_2(x)^2 - \frac{1}{2}\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) - \frac{1}{2}\partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x) + \lambda\Big(\varphi_1(x)\partial_\mu\varphi_1(x) + \varphi_2(x)\partial_\mu\varphi_2(x)\Big)\Big(\varphi_1(x)\partial^\mu\varphi_1(x) + \varphi_2(x)\partial^\mu\varphi_2(x)\Big)\bigg\}. \quad (1)$$

(a) Write the equations of motion for the fields $\varphi_1(x)$ and $\varphi_2(x)$.

(b) Show explicitly that the action S is invariant under the transformation of the fields

$$\varphi_1 \to \cos\alpha\,\varphi_1 + \sin\alpha\,\varphi_2\,,$$
$$\varphi_2 \to -\sin\alpha\,\varphi_1 + \cos\alpha\,\varphi_2\,,$$

where $\alpha = \text{const}$, and derive the Noether current(s) associated with this symmetry.

Hint: In this whole question, it is convenient to first rewrite the action in the form which is manifestly invariant under the given symmetries.

2. (a) Write the general statement of Wick's theorem. Please explain the meaning of each symbol which you use, and in particular give the definition of time ordering and normal ordering of two operators.

Apply Wick's theorem to the following expression

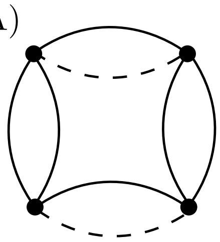
$$\langle 0|T\big(\hat{\phi}_1(x_1)\hat{\phi}_1(x_2)\hat{\phi}_2(x_3)\hat{\phi}_2(x_4)\big)|0\rangle\,,$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are two independent fields.

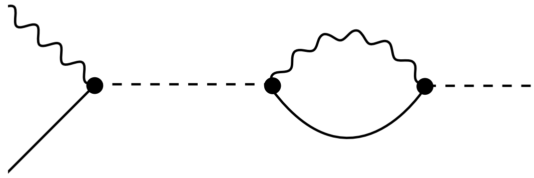
(b) Starting with the expression for $\hat{\phi}_i(x)$ (with $i = 1, 2$) in terms of positive and negative frequency parts $\hat{\phi}_{i,\pm}$ derive the expression which you have written in the previous part. As in the lectures, when doing this you may assume without proof that the commutators which you get give propagators.

3. Figures A and B below show two Feynman graphs which originate from two different theories.

A)



B)



Based on your general knowledge about Feynman graphs answer the following questions:

(a) At which order in perturbation theory do these graphs appear for the first time?

(b) What are the symmetry factors of these graphs?

(c) Write the interaction terms in the Langrangians for the two theories from which these graphs originate. Use standard normalisation for the interactions.

(d) Assume that there are two different theories, such that the only interaction vertex appearing in the first theory is the one in graph A), and the only interaction vertex appearing in the second theory is the one in graph B). For each theory, state whether it is possible for a single incoming particle of one type (species) to change to a single particle of another type (species), if you are considering processes up to and including second order in perturbation theory.

In each case where your answer is yes, justify this by drawing at least one graph which contributes to this process.

4. Consider a relativistic particle whose spacetime coordinates are given in terms of a parameter τ as $X^\mu(\tau)$. The action describing the particle's motion is

$$S[X^\mu(\tau)] = \frac{1}{2}\int d\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2 + i\epsilon\right)$$

(a) Derive the equation of motion of the particle from this action and describe it physically.

(b) Define the amplitude for a particle which is initially at space-time point X_I to be found at X_F to be

$$G(X_F^\mu; X_I^\mu) = \int_0^\infty dT \int \mathcal{D}X^\mu(\tau) \exp\left(\frac{i}{\hbar} S[X^\mu(\tau)]\right)$$

where the path integral is over all paths $X^\mu(\tau)$ with $X^\mu(0) = X_I^\mu$ and $X^\mu(T) = X_F^\mu$. Explain carefully how this can be approximated as

$$G(X_F^\mu; X_I^\mu) \approx \lim_{N\to\infty} \int_0^\infty dT C(N,\Delta) \int d^4X_1 \ldots \int d^4X_{N-1}$$
$$\times \exp \frac{i}{2\hbar} \sum_{k=1}^{N} \left(\frac{1}{\Delta}(X_k - X_{k-1})^\mu (X_k - X_{k-1})_\mu - \Delta(m^2 - i\epsilon)\right) .$$

You should define Δ but need not define $C(N,\Delta)$ here.

(c) Thus obtain the Fourier transform of this amplitude

$$G(P_F; P_I) := \int d^4X_F d^4X_I \, G(X_F; X_I) \, \exp \frac{i}{\hbar}\left(P_I.X_i + P_F.X_F\right) ,$$

defining $C(N,\Delta)$ in such a way as to ensure a finite result. What is this amplitude called in quantum field theory?

Hint: You may use that $\int \exp(-ia(x-y)^2)dx = \sqrt{\pi/ia}$ in your derivation.

5. Consider the generating functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}S[\phi] + \int d^4x J(x)\phi(x)\right)$$

for a free scalar field theory, with action

$$S[\phi] = \frac{1}{2}\int d^4x \, \phi \, (\Box - m^2)\phi \, .$$

(a) Rewrite $Z[J]$ in terms of $\tilde{\phi} = \phi - \int d^4y G(x-y)J(y)$ in order to complete the square in the exponent. What equation must $G(x-y)$ satisfy?

(b) Compute the correlation function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{Z[0]}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z[J]|_{J=0}$$

from $Z[J]$ in terms of $G(x-y)$.

(c) Compute

$$\frac{1}{Z[0]}\frac{1}{3!}\frac{i\lambda}{\hbar}\int d^4y \frac{\delta^3}{\delta J(y)^3}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}Z[J]|_{J=0}$$

from $Z[J]$ (leaving your answer in terms of a single space-time integral involving $G(x-y)$). Draw the corresponding Feynman diagram.

6. The Virasoro generators for the quantum closed relativistic string are given by

$$\hat{L}_m = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} : \hat{\alpha}^\mu_{m-n}\hat{\alpha}^\nu_n : \eta_{\mu\nu}\right) - a\delta_{m,0}\, ,$$

with a similar expression for $\hat{\tilde{L}}_m$. Here $\hat{\alpha}^\mu_0 = \hat{\tilde{\alpha}}^\mu_0 = \sqrt{\alpha'/2}\,\hat{p}^\mu$.

(a) Write down the constraints involving $\hat{L}_m$ and $\hat{\tilde{L}}_m$ and explain their origin in words.

(b) Use the above constraints to show that the mass-squared operator is given in terms of the number operators,

$$\hat{N} := \sum_{n\geq 1}\hat{\alpha}^\mu_{-n}\hat{\alpha}^\nu_n\eta_{\mu\nu}$$

(and similarly for $\tilde{N}$) by

$$M^2 = \frac{2}{\alpha'}\left(\hat{N} + \hat{\tilde{N}} - 2a\right)\, .$$

Also show that there is a constraint

$$0 = \hat{N} - \hat{\tilde{N}}\, .$$

(c) What is the mass of the vacuum and the first excited state (use lightcone gauge)? How can this fix the value of a?

SECTION B

7. An action for two free, real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}m^2\varphi_1^2(x) - \frac{1}{2}\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) - \frac{1}{2}m^2\varphi_2^2(x) - \frac{1}{2}\partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x)\right) \quad (2)$$

$\mu = 0, 1, 2, 3$ and this action is invariant under "rotations"

$$\varphi_1 \to \cos\alpha\,\varphi_1 + \sin\alpha\,\varphi_2\,,$$
$$\varphi_2 \to -\sin\alpha\,\varphi_1 + \cos\alpha\,\varphi_2\,,$$

where $\alpha = \text{const}$.

(a) Write down the classical charge Q associated with this symmetry, as well as the quantum version of this charge.

(b) Work out the normal ordered expression for the quantum charge $\hat{Q}$ found in the previous part.

(c) The normal ordered quantum Hamiltonian $\hat{H}$ for the action (2) is given by

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3}\omega_p \left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\right) \qquad \omega_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

where $\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p}}$ are creation and annihilation operators for the fields φ_1 and φ_2. Using this expression, compute explicitly the commutator $[\hat{H}, \hat{Q}]$. Explain what is the physical meaning of your result.

(d) The quantum momentum operator for the system, in normal-ordered form, is given by

$$\hat{P}_i = \int \frac{d^3p}{(2\pi)^3}p_i \left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\right)\,.$$

What is the value of the commutator $[\hat{P}_i, \hat{Q}]$? You do not need to do this computation explicitly. Can you explain this result?

8. The action for two real scalar fields $\varphi_1(x), \varphi_2(x)$ is given by

$$S = -\int \mathrm{d}^4x \left\{ \frac{1}{2} \left(\sum_{i=1}^{2} \partial_\mu \varphi_i(x) \partial^\mu \varphi_i(x) + \sum_{i=1}^{2} m_i^2 \varphi_i(x)^2 \right) + \lambda \varphi_1(x) \varphi_2(x) + \frac{\lambda}{2} \varphi_1(x)^2 \right\} \quad (3)$$

where λ is a real number, a coupling constant.

(a) Write down the Feynman rules for this theory in position and momentum space. Write the integral expression for the Feynman propagators.

(b) List all the vacuum bubbles which appear in this theory up to and including order λ^2. You should draw all the graphs and write the expressions for these graphs in position space. You do not need to evaluate any of the graphs.

(c) Write the general expression for the Dyson formula for two fields, explain why this formula is very important and what are the limitations of this formula in general. Please explain all the symbols which appear in the formula, and be very precise about the difference between operators and bras and kets on the left- and right-hand side of the formula.

(d) Evaluate the two-point correlators

$$\langle \Omega | T\{\varphi_1(x)\varphi_2(y)\} | \Omega \rangle \quad \text{and} \quad \langle \Omega | T\{\varphi_1(x)\varphi_1(y)\} | \Omega \rangle \,,$$

up to and including second order in perturbation theory. Start by writing the Dyson formula for two fields and show explicitly that all bubble diagrams cancel out.

9. Consider a 0-dimensional "field theory" with action

$$S_\lambda = -\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \,.$$

(a) What does the expression

$$I_\lambda = \frac{\displaystyle\int_{-\infty}^{\infty} \phi^2 \, e^{\frac{i}{\hbar}S_\lambda} \, \mathrm{d}\phi}{\displaystyle\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}S_0} \mathrm{d}\phi}$$

represent physically?

(b) The Fresnel integral is given as

$$\int e^{-ia\phi^2} d\phi = \sqrt{\frac{\pi}{ia}}$$

(which you can assume without proof).
Use this to compute I_0. What does this represent physically?

(c) Similarly compute the order λ term to the series expansion of I_λ.

(d) Compute the $O(\lambda^k)$ term to the series expansion of I_λ for arbitrary k.

(e) Show that the $O(\lambda^k)$ term diverges for large k. Why is perturbation theory still useful?
Hint: You may assume Stirling's formula: $k! \sim k^k$ as $k \to \infty$.

10. Consider the closed string action,

$$S = \int d\tau \int_0^{2\pi} d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \,.$$

(a) Derive the equations of motion and the constraints from the above Lagrangian. **Hint:** You may use without proof that $\frac{\partial}{\partial h^{\alpha\beta}}\sqrt{-h} = -\frac{1}{2}h_{\alpha\beta}\sqrt{-h}$.

(b) In the flat gauge in world-sheet light-cone coordinates, $\sigma^\pm = \tau \pm \sigma$, the world-sheet metric takes the form

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,.$$

Derive the equations of motion and constraints in this gauge.

(c) The generic solution to the equation of motion is given by $X(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-)$ where

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu \frac{1}{n} e^{-in\sigma^-}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \tilde{\alpha}_n^\mu \frac{1}{n} e^{-in\sigma^+} \,.$$

Show that this solution satisfies the equation of motion and the periodicity condition $X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi, \tau)$.

(d) Show that the flat world-sheet metric is invariant under a generic transformation

$$\sigma^+ = f_+(\tilde{\sigma}^+) \,, \quad \sigma^- = f_-(\tilde{\sigma}^-) \,,$$

(where $f_\pm$ are two arbitrary functions) combined with a scale transformation.

(e) Consider the solution where all coefficients $x^\mu, p^\mu, \alpha_n^\mu = 0$, except:

$$p^0 = \frac{k}{\alpha'}, \quad \alpha_1^1 = \tilde{\alpha}_1^1 = -\alpha_{-1}^1 = -\tilde{\alpha}_{-1}^1 = \frac{-i}{\sqrt{2\alpha'}}$$

$$\alpha_1^2 = -\tilde{\alpha}_1^2 = \alpha_{-1}^2 = -\tilde{\alpha}_{-1}^2 = \frac{1}{\sqrt{2\alpha'}} \,,$$

for some constant k. Substitute these values into the general solution, and simplify the result for $X(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-)$ (writing your result in terms of $\cos\tau, \sin\tau, \cos\sigma, \sin\sigma$).

(f) Check this solution satisfies the constraints for some value of k which you should find, and describe this solution physically.

(g) Now consider a different solution where all coefficients $x^\mu, p^\mu, \alpha_n^\mu = 0$, except:

$$p^0 = \frac{k'}{\alpha'}, \quad \alpha_2^1 = \tilde{\alpha}_2^1 = -\alpha_{-2}^1 = -\tilde{\alpha}_{-2}^1 = \frac{-i}{\sqrt{2\alpha'}}$$

$$\alpha_2^2 = -\tilde{\alpha}_2^2 = \alpha_{-2}^2 = -\tilde{\alpha}_{-2}^2 = \frac{1}{\sqrt{2\alpha'}} \,,$$

for some constant k'. Simplify this result as in (*e*), find the value of k' and describe how the motion differs from the previous solution.