Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2019 | MATH4131-WE01 |

| Title: |
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| Probability IV |

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| Time Allowed: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted:<br>Use of electronic calculators is forbidden. |
| Visiting Students may use dictionaries: No | | |

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| Instructions to Candidates: | Credit will be given for:<br>the best **TWO** answers from Section A,<br>the best **THREE** answers from Section B,<br>**AND** the answer to the question in Section C.<br>Questions in Section B and C carry **TWICE** as many marks as those in Section A. |

| Revision: | |
|---|---|

ED01/2019

## SECTION A

1. Let $(S_k)_{k=0}^{2n}$ be a $2n$-step trajectory of a simple symmetric random walk starting at the origin (and making jumps $\pm 1$ with probability $1/2$). Let

$$C_{2n} \stackrel{\mathsf{def}}{=} \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}\,,$$

and consider the probabilities

$$u_{2n} \stackrel{\mathsf{def}}{=} \mathsf{P}(S_{2n}=0)\,, \quad f_{2k} \stackrel{\mathsf{def}}{=} \mathsf{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2k-1} \neq 0, S_{2k}=0)\,.$$

(a) Show that $u_{2n} = \binom{2n}{n} 2^{-2n}$ and $f_{2k} = 2\,C_{2k-2}\,2^{-2k}$.

(b) Deduce that $f_{2k} = \frac{1}{2k}\,u_{2k-2} = u_{2k-2} - u_{2k}$.

(c) Use the result in part b) to show that

$$\mathsf{P}(S_1 \neq 0, \dots, S_{2n} \neq 0) = 1 - \sum_{k=1}^{n} f_{2k} = u_{2n}\,.$$

2. (a) Carefully state the renewal theorem.

(b) Let $(u_n)_{n\geq 0}$ be a sequence defined by $u_0 = 1$ and, for $n > 0$, by $u_n = \sum\limits_{k=1}^{n} f_k u_{n-k}$, where $f_k > 0$ and $\sum_{k=1}^{\infty} f_k \leq 1$.

i. Show that if $\sum_{k=1}^{\infty} \rho^k f_k = 1$ for some $\rho > 0$, then $v_n = \rho^n u_n$, $n \geq 0$, is a renewal sequence generated by a probabilistic collection of weights.

ii. Show that as $n \to \infty$, we have $v_n = \rho^n u_n \to c$, for some constant $c > 0$. Express the value of this constant in terms of the sequence $(f_k)_{k\geq 1}$.

iii. If the constant $\rho$ satisfies $\rho > 1$, deduce that $u_n$ decays to zero exponentially fast. (This improves the $\sum_{k=1}^{\infty} f_k < 1$ claim of the renewal theorem.)

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex real function and let $\xi$ be a random variable with finite mean. Prove Jensen's inequality,

$$\mathsf{E} f(\xi) \geq f\big(\mathsf{E}\xi\big)\,.$$

(b) Let $\xi$ be a random variable with $\mathsf{E}\big(|\xi|^r\big) < \infty$ for some $r > 0$. Prove Lyapunov's inequality,

$$\big(\mathsf{E}(|\xi|^r)\big)^{1/r} \geq \big(\mathsf{E}(|\xi|^s)\big)^{1/s}\,, \qquad \text{where } r > s > 0\,.$$

4. Carefully define order statistics for a sample of i.i.d. random variables.

   Let $X_1$ and $X_2$ be independent $\mathsf{Exp}(1)$ random variables, and let $X_{(1)}$ and $X_{(2)}$ be the corresponding order variables.

   (a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent and find their distributions.

   (b) Compute $\mathsf{E}(X_{(2)} \mid X_{(1)} = x_1)$ and $\mathsf{E}(X_{(1)} \mid X_{(2)} = x_2)$, where $x_1$, $x_2 > 0$.

5. Carefully define the stochastic order $\preccurlyeq$ for random variables.

   (a) Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be Gaussian random variables. In the questions below, justify your answer by proving the result or giving a counter-example.

      i. If $\mu_X \le \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$, is it true that $X \preccurlyeq Y$?

      ii. If $\mu_X = \mu_Y$ and $\sigma_X^2 \le \sigma_Y^2$, is it true that $X \preccurlyeq Y$?

   (b) Let $X \sim \mathsf{Poi}(\lambda)$ and $Y \sim \mathsf{Poi}(\mu)$ be Poisson random variables with $\lambda \le \mu$.

      i. Show that $X \preccurlyeq Y$.

      ii. Show that $\mathsf{E}(X^m) \le \mathsf{E}(Y^m)$ for all $m \ge 0$.

      [Hint: Show that if $Z$ is a random variable with values in $\{0, 1, 2, \dots\}$ and $g(\,\cdot\,) \ge 0$ is an increasing function with $g(0) = 0$, then

$$\mathsf{E}\big(g(Z)\big) = \sum_{k \ge 0} \big(g(k+1) - g(k)\big)\, \mathsf{P}(Z > k)\,. \quad ]$$

6. Let $\pi$ be a permutation of the set $\{1, 2, \dots, n\}$, chosen uniformly at random.

   (a) If $A_m = \{m \text{ is a fixed point of } \pi\}$, find the probability $\mathsf{P}\big(A_{m_1} \cap A_{m_2} \cap \dots \cap A_{m_k}\big)$ for distinct $1 \le m_1 < m_2 < \dots < m_k \le n$.

   (b) Let $S_n$ be the number of fixed points of $\pi$. By using inclusion-exclusion or otherwise, find $\mathsf{P}(S_n > 0) \equiv \mathsf{P}\big(\cup_{m=1}^n A_m\big)$; deduce that $\mathsf{P}(S_n = 0) \to e^{-1}$ as $n \to \infty$.

   (c) Show that, as $n \to \infty$, the distribution of $S_n$ converges to $\mathsf{Poi}(1)$.

## SECTION B

7. For an $n$-sample $\{X_k\}_{k=1}^n$ from the uniform distribution on $[0,1]$, let $X_{(k)}$ and $\Delta_{(k)}X$ be, respectively, the $k$th order variable and the $k$th gap.

   (a) For positive $a$ find the limit $\mathsf{P}(nX_{(1)} > a)$ as $n \to \infty$. What does it tell you about the large-$n$ behaviour of $nX_{(1)} \equiv n\Delta_{(1)}X$?

   (b) By using induction or otherwise, show that
   $$\mathsf{P}\big(\Delta_{(1)}X \ge r_1, \dots, \Delta_{(n+1)}X \ge r_{n+1}\big) = \Big(1 - \sum_{k=1}^{n+1} r_k\Big)^n$$
   if positive $r_k$ satisfy $\sum_{k=1}^{n+1} r_k \le 1$. Deduce that all gaps $\Delta_{(k)}X$ have the same distribution. How big, on average, is the size of the typical gap $\Delta_{(k)}X$ for large $n$?

   (c) Let $\Delta_n^* X = \min_k \Delta_{(k)}X$ be the size of the minimal gap of the $n$-sample under consideration. For positive $a$ find the limit $\mathsf{P}(n^2\Delta_n^* X > a)$ as $n \to \infty$. What does it tell you about the typical size of the minimal gap $\Delta_n^* X$ for large $n$?

8. Let $(X_n)_{n\ge1}$ be independent random variables with common $\mathsf{Exp}(\lambda)$ distribution, $\lambda > 0$.

   (a) Find a constant $c$ such that $\mathsf{P}\Big(\limsup_{n\to\infty} \dfrac{X_n}{\log n} = c\Big) = 1$.

   (b) Define $M_n = \max\limits_{1\le k\le n} X_k$, the running record value at time $n$. Show that with the same constant $c$ as above,
   $$\mathsf{P}\Big(\limsup_{n\to\infty} \frac{M_n}{\log n} = c\Big) = 1\,.$$

   (c) Compute the probability $\mathsf{P}(M_n \le x)$ and use your result to find a constant $c$ such that
   $$\mathsf{P}\Big(\lim_{n\to\infty} \frac{M_n}{\log n} = c\Big) = 1\,.$$

   In your answer you should clearly state every result you use.

   [**Hint:** You may use without proof the following fact: *If $(x_n)_{n\ge1}$ are real numbers, $m_n \equiv \max\limits_{1\le k\le n} x_k$, and a monotone sequence $(b_n)_{n\ge1}$ increases to infinity as $n\to\infty$, then the sets $\{n\in\mathbb{N} : x_n \ge b_n\}$ and $\{n\in\mathbb{N} : m_n \ge b_n\}$ are both finite or both infinite.*]

9. Let $r$ balls be placed randomly and independently into $n$ boxes. Denote by $X_i$ the number of balls in the $i$th box and by $N$ the number of empty boxes.

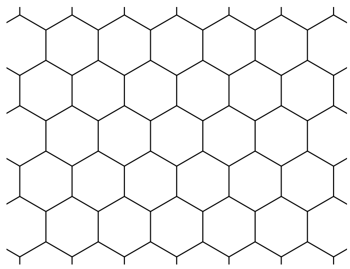
   (a) Show that $\mathsf{E}(n^{-1}N) = \bigl(1 - \frac{1}{n}\bigr)^r$ and $\mathsf{Var}\bigl(n^{-1}N\bigr) \to 0$ as $n \to \infty$.

   (b) Find the fraction of the empty boxes in the limit when $r/n \to c > 0$ as $n \to \infty$.

   (c) Show that $\mathsf{P}(X_1 = k) = \binom{r}{k}(n-1)^{r-k}/n^r$ and identify the limit, as $n \to \infty$, of this probability under the assumption of part (b).

   (d) Find the probability $\mathsf{P}(X_1 = k_1, X_2 = k_2)$; what happens in the limit $n \to \infty$ under the assumption of part (b)?

10. Consider bond percolation on the hexagonal lattice (see the picture below), with every bond independently open with probability $p \in [0, 1]$.



   (a) Carefully define the percolation probability $\theta(p)$; show that it is a non-decreasing function of $p$ and hence define the critical value $p_{\mathsf{c}}$.

   (b) Show that $\theta(p) = 0$ for $p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \geq p'$ for some $p' > 0$.

   (c) Show that $\theta(p) > 0$ for $1 - p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \leq p''$ for some $p'' < 1$.

## SECTION C

11. (a) Carefully state Cramér's theorem on large deviations for sums of independent identically distributed random variables. Show that under the conditions of Cramér's theorem the rate function $I(\,\cdot\,)$ satisfies the inequality $I(a) \geq 0$ for all $a \in \mathbb{R}$; for which values of $a$ does the equality hold?

(b) Let $X_1$, $X_2$,... be independent Bernoulli random variables with

$$\mathsf{P}(X_i = 1) = p\,, \quad \mathsf{P}(X_i = -1) = 1 - p\,, \qquad p \in (0, 1)\,,$$

and let $S_n = X_1 + \cdots + X_n$. Compute the moment generating function $\psi_n(t) \equiv \mathsf{E}e^{tS'_n}$ of the centred sum $S'_n = S_n - \mathsf{E}S_n$ and show that $\psi_n(t) \leq \exp\{nt^2/2\}$ for all real $t$. Deduce the inequality $\mathsf{P}\big(S'_n \geq an\big) \leq \exp\big\{-\frac{na^2}{2}\big\}$, $a > 0$.

[Hint: In your answer you may use without proof the following result:

$$pe^x + (1 - p)e^{-x} \leq \exp\{(2p - 1)x + x^2/2\} \qquad x \in \mathbb{R}\,. \qquad ]$$

(c) In the hard times of a prolonged recession an IT company *Pearsoft* received a takeover bid from one of its competitors. A consultation with shareholders of *Pearsoft* showed that $n_R = 520,000$ of them rejected the offer whereas $n_A = 480,000$ of the shareholders were prepared to accept it. Once the votes were counted, it has been discovered that a virus on the main server of *Pearsoft* independently changed each result to the opposite with probability $r = 0.45$.

i. Let $R$ and $A$ be the actual numbers of rejecting and accepting votes, with total $N = A + R = n_A + n_R = 1,000,000$. Show that, for given $R$, the (random) number $N_R$ of counted rejecting votes has expectation $\mathsf{E}N_R = R + Ar - Rr$; find the moment generating function of the centred variable $N_R - \mathsf{E}N_R$.

ii. Assuming that $2R \leq N$, ie., that the majority of shareholders actually accepted the offer, bound the probability $q_R = \mathsf{P}\big(N_R \geq n_R \mid R \leq N/2\big)$. Should *Pearsoft* repeat the consultation?



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