

# EXAMINATION PAPER

Examination Session: May

2019

Year:

Exam Code:

MATH4141-WE01

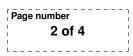
Title:

Geometry IV

Time Allowed:	3 hours				
Additional Material provided:	Formula Sheet				
Materials Permitted:	None				
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.			
Visiting Students may use dictionaries: No					

Instructions to Candidates:	Credit will be given for: the best <b>TWO</b> answers from Section the best <b>THREE</b> answers from Section <b>AND</b> the answer to the question in Section B and C carry <b>T</b> those in Section A.	on B, ection C.	any marks as

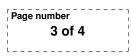
**Revision:** 



### SECTION A

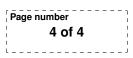
- 1. (a) Let G be a group acting on a set X. Give the definition of the orbit of an element  $x \in X$  under the action of G.
  - (b) Consider the Euclidean plane  $\mathbb{E}^2$  represented by complex numbers. Let G be a group acting on  $\mathbb{E}^2$  and generated by elements g(z) = z + 1 and h(z) = iz. Find the orbit of the point 1 + i under the action of the group G.
- 2. (a) Is it true that affine transformations act transitively on quadrilaterals in  $\mathbb{E}^2$ ? Justify your answer.
  - (b) Let  $A_1A_2A_3$  be a triangle in  $\mathbb{E}^2$ . Denote  $A_4 = A_1$ ,  $A_5 = A_2$ . Let  $B_i$ , i = 1, 2, 3, be a point on the line  $A_iA_{i+1}$  such that  $|B_iA_i| = \frac{1}{2}|A_iA_{i+1}|$  and  $A_i$  lies between  $B_i$  and  $A_{i+1}$ . Similarly, let  $C_i$ , i = 1, 2, 3, be a point on the line  $A_iA_{i+2}$  such that  $|C_iA_i| = \frac{1}{2}|A_iA_{i+2}|$  and  $A_i$  lies between  $C_i$  and  $A_{i+2}$ . Show that the points  $K = B_1B_2 \cap C_1C_3$ ,  $L = B_2B_3 \cap C_2C_3$ ,  $M = B_1B_3 \cap C_1C_2$ are collinear.
- 3. (a) Let ABC be a triangle in  $\mathbb{E}^2$ , and let M and N be the midpoints of AB and AC respectively. Show that  $|MN| = \frac{1}{2}|BC|$ .
  - (b) Let ABC be a triangle in  $S^2$ , and let M and N be the midpoints of AB and AC respectively. Show that  $|MN| > \frac{1}{2}|BC|$ .
- 4. (a) Let  $C_1$ ,  $C_2$ ,  $C_3$  be three mutually tangent circles. Is it always true that there exists a fourth circle  $C_4$  tangent to all three of  $C_1$ ,  $C_2$ ,  $C_3$ ? Justify your answer.
  - (b) Show that any four mutually tangent circles  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  with  $C_1 \cap C_2 \cap C_3 \cap C_4 = \emptyset$  can be taken by a Möbius transformation to some three mutually tangent unit circles inscribed into another circle.
- 5. Let XYZ be an ideal triangle in  $\mathbb{H}^2$ .
  - (a) Show that  $\triangle XYZ$  has an inscribed circle.
  - (b) Find the hyperbolic cosine of the radius of the circle inscribed into  $\triangle XYZ$ .
- 6. (a) Define the angle of parallelism in  $\mathbb{H}^2$ .
  - (b) Let m and n be two orthogonal lines in  $\mathbb{H}^2$ , denote  $O = m \cap n$ . Let  $l_1$  and  $l_3$  be the two distinct lines orthogonal to m and intersecting m at distance a from O. Let  $l_2$  and  $l_4$  be the two distinct lines orthogonal to n and crossing n at distance x from m.

Given a, for which values of x do the lines  $l_1, l_2, l_3, l_4$  compose a quadrilateral having finite area?



## SECTION B

- 7. Let ABC be a triangle in  $\mathbb{H}^2$  (labelled clockwise) with angles  $\alpha, \beta, \gamma$  at A, B, C respectively. Denote by  $R_{X,\varphi}$  a rotation around point X through angle  $\varphi$  in the clockwise direction.
  - (a) Let  $g = R_{A,2\alpha} \circ R_{B,2\beta}$ . Find all the fixed points of g.
  - (b) Find  $h = R_{A,2\alpha} \circ R_{B,2\beta} \circ R_{C,2\gamma}$ . Does it have fixed points?
  - (c) Now, consider  $\varphi = R_{A,\alpha} \circ R_{B,\beta} \circ R_{C,\gamma}$ . Show that  $\varphi$  takes the line AC to itself.
  - (d) How many fixed points has the isometry  $\varphi$  introduced in part (c)? Find the order of  $\varphi.$
- 8. Let A, B, C be points on the unit sphere  $S^2$ . Suppose that  $B \in Pol(A)$  and  $C \in Pol(B)$ .
  - (a) Find  $\angle CAB$ .
  - (b) Suppose that  $\angle CBA = \beta$ . Find the length AC.
  - (c) Let  $\triangle A'B'C'$  be a triangle polar to  $\triangle ABC$ . Given that  $\angle CBA = \beta < \pi/2$ , which of the triangles  $\triangle ABC$  and  $\triangle A'B'C'$  has larger area?
  - (d) Let  $l \subset S^2$  be a line,  $D_1, D_2 \in Pol(l)$  be the two distinct poles to l. Let  $\varepsilon > 0$ , and let  $P_1, P_2$  be two points such that  $d(P_i, l) < \varepsilon$  for i = 1, 2. Given a point  $A \in Pol(P_1P_2)$ , is it true that for at least one of  $D_i$  we have  $d(A, D_i) < \varepsilon$ ?
- 9. (a) Which of the following statements are true? Justify your answer.
  - (i) Projective transformations of  $\mathbb{R}P^2$  act transitively on pairs of projective lines.
  - (ii) Projective transformations of  $\mathbb{R}P^2$  act transitively on triples of projective lines.
  - (b) The points  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_4$  lie on a line a in the Euclidean plane  $\mathbb{E}^2$ , and the points  $B_1$ ,  $B_2$ ,  $B_3$ ,  $B_4$  lie on a line  $b \subset \mathbb{E}^2$ , where a is not parallel to b. Assume that all the four lines  $A_iB_i$ , i = 1, 2, 3, 4 intersect at one point, denote the intersection point by P. Let  $Q = A_1B_2 \cap B_1A_2$  and  $S = A_3B_4 \cap A_4B_3$ . Show that the point  $a \cap b$  lies on the line QS.
  - (c) Assuming that P = (1,0),  $B_1 = (0,0)$ ,  $B_2 = (0,1)$ ,  $B_3 = (0,2)$ ,  $B_4 = (0,3)$ , find the cross-ratio of the lines  $PB_1, PB_2, PB_3, PB_4$ .
  - (d) Formulate the statement dual to the one in part (b).





- 10. The goal of this problem is to prove that if opposite angles of a hyperbolic quadrilateral are equal then there exists a rotation by  $\pi$  taking the quadrilateral to itself, and hence its opposite sides are also equal.
  - (a) Let l be a line in  $\mathbb{H}^2$ , let  $P_1, P_2 \in l$  be points and let  $X \in l \cap \partial \mathbb{H}^2$  be one of the endpoints of l. Let  $S_1, S_2$  be points lying in one half-plane with respect to l such that  $\angle S_1P_1X = \angle S_2P_2X$ . Show that the lines  $S_1P_1$  and  $S_2P_2$  do not intersect.
  - (b) In the assumptions of (a), assume that  $P_1$  lies between X and  $P_2$  on l. Assume that the line m through points  $S_1$  and  $S_2$  is ultra-parallel to l, denote by Y the endpoint of m such that  $S_1$  lies between Y and  $S_2$ . Show that the angle  $\angle YS_1P_1$  is larger than the angle  $\angle YS_2P_2$ .
  - (c) Let ABCD be a quadrilateral in  $\mathbb{H}^2$ . Suppose that  $\angle A = \angle C = \alpha$  and  $\angle B = \angle D = \beta$ . Show that the lines AB and CD are ultra-parallel.
  - (d) In the assumptions of part (c), show that there exists a point O such that a rotation  $R_{O,\pi}$  around O through  $\pi$  takes the quadrilateral ABCD to itself.

### SECTION C

- 11. (a) Define a parabola. Show that a parabola has an axis of symmetry.
  - (b) Given a parabola drawn on the plane, construct a line parallel to the axis of the parabola using ruler and compass. Justify your answer.

(You can use without proofs and clarifications the following constructions:

- the midpoint of a given segment;
- the line perpendicular to a given line through a given point.)
- (c) Given a parabola, ruler and compass, construct the axis of the parabola. Justify your answer.
- (d) On the plane, a parabola is drawn together with a tangent line at a point X of the parabola not lying on the axis. Explain how to construct the focus of the parabola using ruler and compass. Justify your answer.

# Formula sheet

Sine and cosine laws:

	sine law	cosine laws	
$S^2$	$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$	$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$ $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$	
$\mathbb{E}^2$	$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$	$a^2 = b^2 + c^2 - 2bc\cos\alpha$	
$\mathbb{H}^2$	$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$	$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$ $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a$	

<u>Circles:</u>

	$S^2$	$\mathbb{E}^2$	$\mathbb{H}^2$
Circumference of a circle	$2\pi\sin R$	$2\pi R$	$2\pi\sinh R$
Area of a disc	$4\pi\sin^2(\frac{R}{2})$	$\pi R^2$	$4\pi\sinh^2(\frac{R}{2})$

Angle of parallelism in hyperbolic geometry:

For a point on distance a from the line, the angle of parallelism  $\varphi$  satisfies

$$\sin\varphi = \frac{1}{\cosh a}$$

Distance formula in the upper half-plane model of hyperbolic geometry:

$$\cosh d(u, v) = 1 + \frac{|u - v|^2}{2\operatorname{Im}(u)\operatorname{Im}(v)}$$

<u>Distance formula</u> in the hyperboloid model of hyperbolic geometry: For  $u, v \in \mathbb{R}^{2,1}$ , let  $Q = |\frac{(u,v)^2}{(u,u)(v,v)}|$ . Then

 $\begin{array}{ll} \mathrm{if}\ (u,u)<0,\ (v,v)<0 \ \ \mathrm{then} \ \ Q=\cosh^2 d(pt,pt) \\ \\ \mathrm{if}\ (u,u)<0,\ (v,v)>0 \ \ \mathrm{then} \ \ Q=\sinh^2 d(pt,line) \\ \\ \mathrm{if}\ (u,u)>0,\ (v,v)>0 \ \ \mathrm{then} \ \ Q<1\Rightarrow \mathrm{intersecting\ lines}, \ \ Q=\cos^2\alpha; \\ \\ Q=1\Rightarrow \mathrm{parallel\ lines}; \\ Q>1\Rightarrow \mathrm{ultraparallel\ lines}, \ Q=\cosh^2 d(line,line) \\ \end{array}$