Durham University

EXAMINATION PAPER

Examination Session: May	**Year:** 2019	**Exam Code:** MATH4141-WE01

Title: Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	Formula Sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let G be a group acting on a set X. Give the definition of the orbit of an element $x \in X$ under the action of G.

   (b) Consider the Euclidean plane $\mathbb{E}^2$ represented by complex numbers. Let G be a group acting on $\mathbb{E}^2$ and generated by elements $g(z) = z + 1$ and $h(z) = iz$. Find the orbit of the point $1 + i$ under the action of the group G.

2. (a) Is it true that affine transformations act transitively on quadrilaterals in $\mathbb{E}^2$? Justify your answer.

   (b) Let $A_1A_2A_3$ be a triangle in $\mathbb{E}^2$. Denote $A_4 = A_1$, $A_5 = A_2$. Let B_i, $i = 1, 2, 3$, be a point on the line A_iA_{i+1} such that $|B_iA_i| = \frac{1}{2}|A_iA_{i+1}|$ and A_i lies between B_i and A_{i+1}. Similarly, let C_i, $i = 1, 2, 3$, be a point on the line A_iA_{i+2} such that $|C_iA_i| = \frac{1}{2}|A_iA_{i+2}|$ and A_i lies between C_i and A_{i+2}.

   Show that the points $K = B_1B_2 \cap C_1C_3$, $L = B_2B_3 \cap C_2C_3$, $M = B_1B_3 \cap C_1C_2$ are collinear.

3. (a) Let ABC be a triangle in $\mathbb{E}^2$, and let M and N be the midpoints of AB and AC respectively. Show that $|MN| = \frac{1}{2}|BC|$.

   (b) Let ABC be a triangle in S^2, and let M and N be the midpoints of AB and AC respectively. Show that $|MN| > \frac{1}{2}|BC|$.

4. (a) Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be three mutually tangent circles. Is it always true that there exists a fourth circle $\mathcal{C}_4$ tangent to all three of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$? Justify your answer.

   (b) Show that any four mutually tangent circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ with $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \cap \mathcal{C}_4 = \emptyset$ can be taken by a Möbius transformation to some three mutually tangent unit circles inscribed into another circle.

5. Let XYZ be an ideal triangle in $\mathbb{H}^2$.

   (a) Show that $\triangle XYZ$ has an inscribed circle.

   (b) Find the hyperbolic cosine of the radius of the circle inscribed into $\triangle XYZ$.

6. (a) Define the angle of parallelism in $\mathbb{H}^2$.

   (b) Let m and n be two orthogonal lines in $\mathbb{H}^2$, denote $O = m \cap n$. Let l_1 and l_3 be the two distinct lines orthogonal to m and intersecting m at distance a from O. Let l_2 and l_4 be the two distinct lines orthogonal to n and crossing n at distance x from m.

   Given a, for which values of x do the lines l_1, l_2, l_3, l_4 compose a quadrilateral having finite area?

SECTION B

7. Let ABC be a triangle in $\mathbb{H}^2$ (labelled clockwise) with angles α, β, γ at A, B, C respectively. Denote by $R_{X,\varphi}$ a rotation around point X through angle φ in the clockwise direction.

   (a) Let $g = R_{A,2\alpha} \circ R_{B,2\beta}$. Find all the fixed points of g.

   (b) Find $h = R_{A,2\alpha} \circ R_{B,2\beta} \circ R_{C,2\gamma}$. Does it have fixed points?

   (c) Now, consider $\varphi = R_{A,\alpha} \circ R_{B,\beta} \circ R_{C,\gamma}$. Show that φ takes the line AC to itself.

   (d) How many fixed points has the isometry φ introduced in part (c)? Find the order of φ.

8. Let A, B, C be points on the unit sphere S^2. Suppose that $B \in Pol(A)$ and $C \in Pol(B)$.

   (a) Find $\angle CAB$.

   (b) Suppose that $\angle CBA = \beta$. Find the length AC.

   (c) Let $\triangle A'B'C'$ be a triangle polar to $\triangle ABC$. Given that $\angle CBA = \beta < \pi/2$, which of the triangles $\triangle ABC$ and $\triangle A'B'C'$ has larger area?

   (d) Let $l \subset S^2$ be a line, $D_1, D_2 \in Pol(l)$ be the two distinct poles to l. Let $\varepsilon > 0$, and let P_1, P_2 be two points such that $d(P_i, l) < \varepsilon$ for $i = 1, 2$. Given a point $A \in Pol(P_1P_2)$, is it true that for at least one of D_i we have $d(A, D_i) < \varepsilon$?

9. (a) Which of the following statements are true? Justify your answer.

   (i) Projective transformations of $\mathbb{R}P^2$ act transitively on pairs of projective lines.

   (ii) Projective transformations of $\mathbb{R}P^2$ act transitively on triples of projective lines.

   (b) The points A_1, A_2, A_3, A_4 lie on a line a in the Euclidean plane $\mathbb{E}^2$, and the points B_1, B_2, B_3, B_4 lie on a line $b \subset \mathbb{E}^2$, where a is not parallel to b. Assume that all the four lines A_iB_i, $i = 1, 2, 3, 4$ intersect at one point, denote the intersection point by P. Let $Q = A_1B_2 \cap B_1A_2$ and $S = A_3B_4 \cap A_4B_3$. Show that the point $a \cap b$ lies on the line QS.

   (c) Assuming that $P = (1, 0)$, $B_1 = (0, 0)$, $B_2 = (0, 1)$, $B_3 = (0, 2)$, $B_4 = (0, 3)$, find the cross-ratio of the lines PB_1, PB_2, PB_3, PB_4.

   (d) Formulate the statement dual to the one in part (b).

10. The goal of this problem is to prove that if opposite angles of a hyperbolic quadrilateral are equal then there exists a rotation by π taking the quadrilateral to itself, and hence its opposite sides are also equal.

    (a) Let l be a line in $\mathbb{H}^2$, let $P_1, P_2 \in l$ be points and let $X \in l \cap \partial\mathbb{H}^2$ be one of the endpoints of l. Let S_1, S_2 be points lying in one half-plane with respect to l such that $\angle S_1P_1X = \angle S_2P_2X$. Show that the lines S_1P_1 and S_2P_2 do not intersect.

    (b) In the assumptions of (a), assume that P_1 lies between X and P_2 on l. Assume that the line m through points S_1 and S_2 is ultra-parallel to l, denote by Y the endpoint of m such that S_1 lies between Y and S_2. Show that the angle $\angle YS_1P_1$ is larger than the angle $\angle YS_2P_2$.

    (c) Let $ABCD$ be a quadrilateral in $\mathbb{H}^2$. Suppose that $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$. Show that the lines AB and CD are ultra-parallel.

    (d) In the assumptions of part (c), show that there exists a point O such that a rotation $R_{O,\pi}$ around O through π takes the quadrilateral $ABCD$ to itself.

SECTION C

11. (a) Define a parabola. Show that a parabola has an axis of symmetry.

    (b) Given a parabola drawn on the plane, construct a line parallel to the axis of the parabola using ruler and compass. Justify your answer.

    (You can use without proofs and clarifications the following constructions:

    - *the midpoint of a given segment;*
    - *the line perpendicular to a given line through a given point.)*

    (c) Given a parabola, ruler and compass, construct the axis of the parabola. Justify your answer.

    (d) On the plane, a parabola is drawn together with a tangent line at a point X of the parabola not lying on the axis. Explain how to construct the focus of the parabola using ruler and compass. Justify your answer.

Formula sheet

Sine and cosine laws:

	sine law	cosine laws
S^2	$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$	$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$ $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$
$\mathbb{E}^2$	$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$	$a^2 = b^2 + c^2 - 2bc \cos \alpha$
$\mathbb{H}^2$	$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$	$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$ $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a$

Circles:

	S^2	$\mathbb{E}^2$	$\mathbb{H}^2$
Circumference of a circle	$2\pi \sin R$	$2\pi R$	$2\pi \sinh R$
Area of a disc	$4\pi \sin^2(\frac{R}{2})$	πR^2	$4\pi \sinh^2(\frac{R}{2})$

Angle of parallelism in hyperbolic geometry:
For a point on distance a from the line, the angle of parallelism φ satisfies

$$\sin \varphi = \frac{1}{\cosh a}$$

Distance formula in the upper half-plane model of hyperbolic geometry:

$$\cosh d(u,v) = 1 + \frac{|u-v|^2}{2\mathrm{Im}(u)\mathrm{Im}(v)}$$

Distance formula in the hyperboloid model of hyperbolic geometry:
For $u, v \in \mathbb{R}^{2,1}$, let $Q = |\frac{(u,v)^2}{(u,u)(v,v)}|$. Then

if $(u,u) < 0$, $(v,v) < 0$ then $Q = \cosh^2 d(pt, pt)$

if $(u,u) < 0$, $(v,v) > 0$ then $Q = \sinh^2 d(pt, line)$

if $(u,u) > 0$, $(v,v) > 0$ then $Q < 1 \Rightarrow$ intersecting lines, $Q = \cos^2 \alpha$;
$Q = 1 \Rightarrow$ parallel lines;
$Q > 1 \Rightarrow$ ultraparallel lines, $Q = \cosh^2 d(line, line)$