Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4151-WE01

Title:
Topics in Algebra & Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) State the convergence lemma with respect to a lattice Ω in $\mathbb{C}$.

   (b) For $k \geq 3$, let
   $$f_k(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^k}.$$
   Use the Weierstrass M-test to show that f_k is a meromorphic function on $\mathbb{C}$.

   (c) Identify f_3 in terms of the Weierstrass theory of elliptic functions. (No justification required.)

2. State and prove Liouville's Theorem B for elliptic functions for a lattice Ω in $\mathbb{C}$. Explain how this implies Liouville's Theorem C.

3. (a) Consider the function $f_k(z) = f_k(z, \Omega)$ from Question 1. For $\alpha \neq 0$ show that
   $$f_k(z; \alpha\Omega) = \alpha^{-k} f_k\left(\frac{z}{\alpha}; \Omega\right).$$

   (b) Let $f_k(z;\tau) := f_k(z;\Omega_\tau)$ for $\Omega_\tau = \mathbb{Z}\tau + \mathbb{Z}1$ with $\tau \in \mathbb{H}$, the upper half plane. Explain that we have
   $$\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d) = \Omega_\tau$$
   for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and conclude using (a)
   $$f_k\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f_k(z;\tau).$$

4. (a) Consider S and T, the standard generators of $\mathrm{SL}_2(\mathbb{Z})$. Find the fixed points of S, TS, and TST in the upper half plane, if any. (Justify your findings.)

   (b) Let $f \in M_k$, where M_k denotes the space of holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. By using (a), or otherwise, show that
   $$f(i) f\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) = 0 \qquad \text{for } k \not\equiv 0 \pmod{12}.$$

5. (a) Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ be the Fourier expansion of a normalized Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. Write down the Euler product of its associated L-function.

   (b) Write $\tau(p^3)$, for p a prime, as a polynomial in $\tau(p)$, where $\tau(n)$ denotes Ramanujan's τ-function. Use your expression to explicitly determine $\tau(8)$.

6. (a) Let $\zeta(s)$ be the Riemann zeta function. Define what is meant by the completed Riemann zeta function, often denoted $Z(s)$, and give its functional equation.

   (b) Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_{14}$, $f \neq 0$, where M_k denotes the space of holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k.
   Show that the quotient $\frac{a_2}{a_1}$ is independent of $f \in M_{14}$, and determine its value. Carefully state what results you use.

SECTION B

7. (a) State Liouville's Theorem D (Abel's relation) for elliptic functions with respect to a lattice Ω in $\mathbb{C}$.

   (b) For $u, v \in \mathbb{C}$ fixed with $u \not\equiv \pm v \mod \Omega$, consider the function

$$f(z) = \det \begin{pmatrix} 1 & \wp(z) & \wp'(z) \\ 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \end{pmatrix},$$

   where $\wp$ denotes the Weierstrass $\wp$-function with respect to Ω. Show that f is an elliptic function. Find its order by considering the poles of f. State explicitly where you use the hypothesis on u and v.

   (c) Show that $z = u$ and $z = v$ are zeros of f. Conclude that $-(u + v)$ is also a zero of f.

   (d) Use the associated differential equation/elliptic curve to derive from (c) the analytic form of the addition law for $\wp(z)$. (You may use that the above determinant vanishes exactly when the points $(\wp(z), \wp'(z))$, $(\wp(u), \wp'(u))$, $(\wp(v), \wp'(v))$ are collinear in $\mathbb{C}^2$.)

8. Let Ω be a lattice in $\mathbb{C}$ with basis $\{\omega_1, \omega_2\}$ and consider the Weierstrass functions $\wp(z)$ and $\sigma(z)$ with respect to Ω.

   (a) State the 'representation theorem' for elliptic functions with respect to the σ-function.

   (b) Consider the function

$$f(z) := \wp(z) - \wp\left(\frac{\omega_1}{2}\right).$$

   Find the zeros, poles, and the order of f. Use this to express f in terms of σ. As a consequence find explicitly a meromorphic function g such that $f(z) = (g(z))^2$, that is, f is the square of g. (Warning: "$f^{1/2}$" is not a correct answer.)

   (c) Give a one-line argument why g as in (b) cannot be an elliptic function for Ω.

   (d) Use the transformation property of the σ-function, given for $z \in \mathbb{C}$ and $\omega \in \Omega$ by $\sigma(z + \omega) = \chi(\omega) e^{\eta(\omega)(z+\omega/2)} \sigma(z)$ with $\chi(\omega) = \pm 1$, to show that g as in (b) is elliptic for the (sub-)lattice $\Omega' = \mathbb{Z}\omega_1 + \mathbb{Z}(2\omega_2)$. What is the order of g (with respect to Ω')?

9. Let k and ℓ be even integers and let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ and $g(\tau) = \sum_{n=0}^{\infty} b_n q^n$ be two modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k and ℓ, respectively.

   (a) Differentiate the transformation equation for f with respect to $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ to arrive at a formula expressing $f'(\gamma\tau)$ in terms of $f'(\tau)$, $j(\gamma, \tau)$ and $f(\tau)$.

   (b) For f and g as above show that their bracket $[f, g]$, defined by

   $$[f, g] := kfg' - \ell f'g,$$

   is also a modular form for $\mathrm{SL}_2(\mathbb{Z})$. What is the weight of $[f, g]$?

   (c) Compute the Fourier expansion of $[f, g]$ in terms of the ones for f and g. In particular, show that $[f, g]$ is in fact a cusp form.

   (d) Express the Fourier coefficients of the bracket $[E_4, E_6]$ of the Eisenstein series $E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$ and $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$ in terms of Ramanujan's τ-function.

10. Let $f(\tau) = E_6^4(\tau) - 2E_6^2(\tau)E_4^3(\tau) + E_4^6(\tau)$, where E_4 and E_6 are the Eisenstein series defined in Question 9.

    (a) Show that $f \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, the vector space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight 24.

    (b) Let $\Delta(\tau)$ be the discriminant function. Show that the dimension of $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ equals 2, and show that $g_1(\tau) := \Delta^2(\tau)$ and $g_2(\tau) := \Delta(\tau)E_4^3(\tau)$ together form a basis for it. (Carefully state all the results that you are using.)

    (c) Put $\tilde{f}(\tau) = 12^{-6} f(\tau)$ and express $\tilde{f}(\tau) = \sum_{n \geq 1} a_n q^n$ in terms of the basis given in (b). Furthermore, determine explicitly a_n for $n = 1, 2, 3, 4$.

    (d) Compute the action of the Hecke operator T_2 on $\tilde{f}$ as defined in part (c) and express $T_2\tilde{f}$ in terms of the basis $\{g_1, g_2\}$ from part (b).