Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4161-WE01

Title:
Algebraic Topology IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. (a) Let $f\colon X \to Y$ be a map. Define C_f, the *mapping cone* of f and define a map $g\colon Y \to C_f$ which makes
$$X \xrightarrow{f} Y \xrightarrow{g} C_f$$
a cofibration sequence (you do not need to prove it is a cofibration).

   (b) Prove that the composite $gf\colon X \to C_f$ is homotopic to a constant map.

   (c) Prove that if $h\colon Y \to Z$ has composite hf homotopic to a constant map, then there is a map $k\colon C_f \to Z$ with $h = kg$.

2. (a) Let C_* and D_* be chain complexes of R-modules for some ring R. Define what is meant by a *chain map* $f\colon C_* \to D_*$.

   (b) For $n \in \mathbb{N}$, explain how a chain map of chain complexes $f\colon C_* \to D_*$ gives a homomorphism $f_*\colon H_n(C) \to H_n(D)$. (You may assume anything you need about the definition of the homology of a chain complex. You should show that your f_* is well defined, but you do not need to show that it is a homomorphism.)

   (c) Prove that degree n homology $H_n(-)$ is a functor from chain complexes to R-modules,

3. For each of the following, say whether you think the statement is true or false. If you think it true, prove it; if you think it false, give a counter-example. You may quote without proving any results from lectures.

   (a) If $f\colon X \to Y$ is the inclusion of a subspace X of Y, then $f_*\colon H_n(X) \to H_n(Y)$ is an injection.

   (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $B \cong A \oplus C$.

   (c) Let $X = \{x \in \mathbb{R}^2\,,\ \|x\| < 1\}$. Then any map $f\colon X \to X$ has a fixed point.

4. (a) Write down the cohomology rings of the spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$, without proof.

   (b) Show that these two spaces are not homotopy equivalent.

5. Let Σ_g be the closed orientable surface of genus g. Show that there is a degree one map $\Sigma_g \to \Sigma_h$ if and only if $g \geq h$.

6. Let $f\colon S^4 \to S^2 \times S^2$ be a map. Prove that f is degree zero.

SECTION B

7. (a) State and construct the *Mayer-Vietoris Exact Sequence* for simplicial chain complexes. You should state, but may assume without proof, the *Snake Lemma*.

   (b) In this part, assume all spaces are triangulable and all maps can be realised as simplicial maps. You may assume without proof the homology of the circle S^1. Suppose the space $X = A \cup B$ where the subspaces A, B and $A \cap B$ are all homotopic to S^1. Prove, using the Mayer-Vietoris sequence or otherwise, that if $H_2(X) \neq 0$ then $H_1(X)$ is a free abelian group.

   (c) As in the previous part, the triangulable spaces X here have decompositions $X = A \cup B$ where the subspaces A, B and $A \cap B$ are all homotopic to S^1. In the following, you just need to give the examples without justification.

      (i) Give an example of X and decomposition $A \cup B$ where $H_2(X) \neq 0$.

      (ii) Give an example of X and decomposition $A \cup B$ where $H_1(X)$ has a torsion element of order 2.

      (iii) Give an example of X and decomposition $A \cup B$ where $H_1(X)$ has a torsion element of order 3.

8. (a) State and prove the *Borsuk-Ulam Theorem.* You may assume without proof that, for $n > 0$, any map $g\colon S^n \to S^n$ that satisfies $g(-x) = -g(x)$ has odd degree.

   (b) State and prove the *Ham Sandwich Theorem* in $\mathbb{R}^3$.

9. (a) State the Künneth short exact sequence for cohomology.

   (b) Show that $\mathrm{Tor}_1(H^i(\mathbb{CP}^2;\mathbb{Z}), H^j(S^2;\mathbb{Z})) = 0$ for all i, j. You may state the cohomology groups of $\mathbb{CP}^2$ and S^2 without proof.

   (c) It follows from (b) that the left hand map of the Künneth sequence is an isomorphism of graded rings. Compute the cohomology groups of $\mathbb{CP}^2 \times S^2$ with $\mathbb{Z}$ coefficients, and describe explicit generators.

   (d) Compute the cohomology ring of $\mathbb{CP}^2 \times S^2$. It is enough to give the products of your explicit generators from (c) in a multiplication table.

10. (a) Define the Euler characteristic of a closed, connected 4-dimensional manifold M in terms of its homology.

   (b) Suppose that $\pi_1(M) = \{1\}$, so in particular $H_1(M;\mathbb{Z}) = 0$. Show that $H_2(M;\mathbb{Z})$ is free abelian.

   (c) Suppose that $\chi(M) = 6$. Compute $H_j(M;\mathbb{Z})$ and $H^j(M;\mathbb{Z})$ for all j.