EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4171-WE01

Title: Riemannian Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the standard two-dimensional sphere, let $\mathbb{R}P^2$ be the real projective plane and $\pi : S^2 \to \mathbb{R}P^2$ be the canonical projection, that is, $\pi(x) = \mathbb{R}x$. Let

$$c : (-\varepsilon, \varepsilon) \to S^2, \quad c(t) = (\frac{1}{2}\cos(t), \frac{1}{2}\sin(t), \frac{\sqrt{3}}{2})$$

and

$$f : \mathbb{R}P^2 \to \mathbb{R}, \quad f(\mathbb{R}(z_1, z_2, z_3)) = \frac{(z_1 + z_2 + z_3)^2}{z_1^2 + z_2^2 + z_3^2}.$$

(a) Let $\gamma = \pi \circ c$. Calculate $\gamma'(0)(f)$.

(b) Let $\varphi = (x_1, x_2) : U \to \mathbb{R}^2$ with $U = \{\mathbb{R}(z_1, z_2, z_3) \mid z_1 \neq 0\} \subset \mathbb{R}P^2$ be the following coordinate chart of $\mathbb{R}P^2$:

$$\varphi(\mathbb{R}(z_1, z_2, z_3)) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right).$$

Express $\gamma'(t)$ in the form

$$\alpha_1(t)\frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \alpha_2(t)\frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$

2. Let (M, g) be a Riemannian manifold of non-positive sectional curvature.

(a) Give the definition of a Jacobi field.

(b) Let $J : [a, b] \to TM$ be a Jacobi field along a geodesic $c : [a, b] \to M$. Show that non-positive sectional curvature implies

$$\frac{d}{dt}\langle \frac{D}{dt}J, J\rangle \geq 0.$$

(c) Show that non-positive sectional curvature of M implies that M has no conjugate points.

3. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + 2y^2\}$ and X be a smooth vector field on $\mathbb{R}^3$ given by

$$X(x, y, z) = (1 + 2y^2, -xy, 2x).$$

(a) Show that the restriction of X to the manifold M is a vector field tangential to M.

(b) A coordinate chart $\varphi : U \to V = (0, 2\pi) \times (0, \infty)$ of M is given by

$$\varphi^{-1} : (0, 2\pi) \times (0, \infty) \to M, \quad \varphi^{-1}(\alpha, h) = (h\cos\alpha, \frac{1}{\sqrt{2}}h\sin\alpha, h^2).$$

Calculate $f, g : V \to \mathbb{R}$ explicitly such that

$$X(\varphi^{-1}(\alpha, h)) = f(\alpha, h)\frac{\partial}{\partial \alpha}|_{\varphi^{-1}(\alpha,h)} + g(\alpha, h)\frac{\partial}{\partial h}|_{\varphi^{-1}(\alpha,h)}.$$

4. Let $\mathbb{H}^2 = \{z = x + yi \in \mathbb{C} \mid y > 0\}$ be the upper half plane model of the hyperbolic plane with metric

$$g_z(v, w) = \frac{\langle v, w\rangle}{y^2}, \quad \text{for all } v, w \in T_z\mathbb{H}^2,$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. Let $c : [a, b] \to \mathbb{H}^2$, $c(t) = ti$.

(a) Calculate the length $L(c)$.

(b) Let $\gamma : [\alpha, \beta] \to \mathbb{H}^2$ be another smooth curve with $\gamma(\alpha) = ai$ and $\gamma(\beta) = bi$. Show that $L(\gamma) \geq L(c)$.

5. Let G be a Lie group and $L_g : G \to G$ be defined by $L_g(h) = gh$ for all $g, h \in G$. Recall that a vector field X on G is called left invariant if $DL_g \circ X = X \circ L_g$ for all $g \in G$, that is, for all $g, h \in G$,

$$DL_g(h)(X(h)) = X(L_g(h)) = X(gh).$$

(a) Let $F : M \to N$ be a smooth map between the manifolds M, N, $v \in T_pM$ and $f \in C^\infty(N)$. Show that we have

$$(DF(p)(v))(f) = v(f \circ F).$$

(b) Let X be a left invariant vector field on G. Using (a), show that we have the following identity in $C^\infty(G)$ for all $g \in G$ and all $f \in C^\infty(G)$:

$$X(f \circ L_g) = (Xf) \circ L_g.$$

6. (a) Give the definition of an affine connection ∇ on a smooth manifold.

(b) Let ∇^1 be an affine connection on a smooth manifold M. Show that ∇^2, defined by

$$\nabla^2_X Y = [X, Y] + \nabla^1_Y X$$

for any pair of smooth vector fields X, Y on M, is also an affine connection.

SECTION B

7. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Let $\widehat{g}$ be a second Riemannian metric on M given by $\widehat{g} = e^f \cdot g$. Let ∇ and $\widehat{\nabla}$ be the Levi-Civita connections associated to the metrics g and $\widehat{g}$, respectively.

   (a) Let $\varphi = (x_1, \ldots, x_n) : U \to V$ (with $U \subset M$ and $V \subset \mathbb{R}^n$ open) be a local coordinate chart of M. Show that

$$\widehat{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \frac{1}{2}\left(\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}\right) - \frac{1}{2} g_{ij} \sum_k \left(\sum_l g^{kl} \frac{\partial f}{\partial x_l}\right) \frac{\partial}{\partial x_k}.$$

   (b) Let $M = \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the identity map as a global coordinate system. Let $g, \widehat{g}$ be the following two Riemannian metrics on M: $g_{ij} = \delta_{ij}$, that is, g is the standard Euclidean metric and $\widehat{g}_{ij} = \frac{1}{y^2} g_{ij}$, that is, $\widehat{g}$ is the hyperbolic plane metric. Use (a) to calculate

$$\widehat{\nabla}_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}, \quad \widehat{\nabla}_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \quad \widehat{\nabla}_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}, \quad \widehat{\nabla}_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}.$$

   Hint: You may use without proof that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = 0.$$

   (c) Let (M, g) be a Riemannian manifold and $\widehat{g} = C \cdot g$ with $C > 0$. Use (a) to show that

$$\widehat{K}(\Sigma_p) = \frac{1}{C} K(\Sigma_p),$$

   where K and $\widehat{K}$ denote the sectional curvatures of a 2-plane $\Sigma_p \subset T_pM$ with respect to the metrics g and $\widehat{g}$, respectively.

8. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ and $\varphi^{-1}(\alpha, \beta) = (\sin\alpha\sin\beta, \sin\alpha\cos\beta, \cos\alpha)$, $\alpha \in (0, \pi)$, $\beta \in [0, 2\pi]$ be a coordinate chart. Fix $\alpha = \pi/6$ and consider the closed curve

$$c(t) = \varphi^{-1}(\pi/6, t), \qquad t \in [0, 2\pi],$$

and the vector field

$$X(t) = a(t)\frac{\partial}{\partial\alpha}|_{c(t)} + b(t)\frac{\partial}{\partial\beta}|_{c(t)}$$

along c.

(a) Calculate the first fundamental form g_{ij} of this coordinate chart and show that

$$\nabla_{\frac{\partial}{\partial\alpha}}\frac{\partial}{\partial\alpha} = 0, \qquad \nabla_{\frac{\partial}{\partial\alpha}}\frac{\partial}{\partial\beta} = \nabla_{\frac{\partial}{\partial\beta}}\frac{\partial}{\partial\alpha} = \frac{\cos\alpha}{\sin\alpha}\frac{\partial}{\partial\beta}, \qquad \nabla_{\frac{\partial}{\partial\beta}}\frac{\partial}{\partial\beta} = -\sin\alpha\cos\alpha\frac{\partial}{\partial\alpha}.$$

(b) Calculate the solution of the equation

$$\frac{D}{dt}X(t) = X(t), \qquad X(0) = c'(0).$$

Hint: You may use $\sin(\pi/6) = 1/2$ and without proof that the solution of

$$\begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \begin{pmatrix} 1 & c_1 \\ -c_2 & 1 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \qquad \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

for $c_1, c_2 > 0$ is given by

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} e^t\cos(\sqrt{c_1c_2}t) & e^t\sqrt{\frac{c_1}{c_2}}\sin(\sqrt{c_1c_2}t) \\ -e^t\sqrt{\frac{c_2}{c_1}}\sin(\sqrt{c_1c_2}t) & e^t\cos(\sqrt{c_1c_2}t) \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

9. (a) Let $SL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det(A) = 1\}$. Show that

$$T_{\text{Id}}SL(n,\mathbb{R}) = \{C \in M(n,\mathbb{R}) \mid \text{tr}(C) = 0\}.$$

Hint: You may use without proof that, for every smooth curve $A : (-\epsilon, \epsilon) \to GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det(A) \neq 0\}$, we have:

$$(\det A)'(t) = (\det A(t)) \cdot \text{tr}(A^{-1}(t)A'(t))$$

and that $\dim SL(n,\mathbb{R}) = n^2 - 1$.

(b) Let G be a Lie group with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and X, Y, Z be left invariant vector fields on G. Derive from the identity

$$\nabla_X Y = \frac{1}{2}[X,Y], \tag{1}$$

that

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

Hint: You may use (1), Jacobi's identity and the fact that Lie brackets of left invariant vector fields are again left invariant without proof.

(c) Assume that X, Y are orthonormal. Using (b), show that the sectional curvature of the planes Σ_h, spanned by $X(h), Y(h) \in T_hG$, is given by

$$K(\Sigma_h) = \frac{1}{4}\|[X,Y]\|^2.$$

(d) The tangent space $T_{\text{Id}}O(n) = \{X \in M(n,\mathbb{R}) \mid X^\top = -X\}$ can be canonically identified with the Lie algebra of left invariant vector fields on $O(n)$. We define a bi-invariant metric $\langle \cdot, \cdot \rangle$ on $O(n)$ via

$$\langle X, Y \rangle = \frac{1}{2}\text{tr}(X^\top Y)$$

and the Lie bracket by

$$[X,Y] = XY - YX.$$

Show that

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_{\text{Id}}O(3) \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in T_{\text{Id}}O(3)$$

are orthonormal in $T_{\text{Id}}O(3)$ and calculate the sectional curvature of the planes spanned by X, Y.

10. Let $M = \mathbb{R}^2$ be equipped with the Riemannian metric

$$g_{(x,y)}(v,w) = \frac{4}{(1+x^2+y^2)^2}\langle v,w\rangle \qquad \text{for all } v,w \in T_{(x,y)}M,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product.

(a) Let $f : S^2\backslash\{(0,0,1)\} \to M$ be the stereographic projection from the standard unit sphere $S^2 \subset \mathbb{R}^3$. That is: For $(z_1,z_2,z_3) \in S^2\backslash\{(0,0,1)\}$, let $(x,y,0) \in \mathbb{R}^3$ be the intersection point of the straight Euclidean line through $(0,0,1)$ and (z_1,z_2,z_3) with the horizontal coordinate plane. Then we define $f(z_1,z_2,z_3) = (x,y)$. Show that

$$f(z_1,z_2,z_3) = \left(\frac{z_1}{1-z_3}, \frac{z_2}{1-z_3}\right).$$

(b) Show that the inverse map f^{-1} is given by

$$f^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

(c) Let $z = (z_1,z_2,z_3) \in S^2\backslash\{(0,0,1)\}$ and $v = (v_1,v_2,v_3) \in T_zS^2\backslash\{(0,0,1)\}$. Show that

$$Df(z)(v) = \frac{1}{(1-z_3)^2}\left((1-z_3)v_1 + z_1v_3, (1-z_3)v_2 + z_2v_3\right).$$

Let $z = (1,0,0)$, $v = (0,v_2,v_3) \in T_zS^2\backslash\{(0,0,1)\}$ and $(x,y) = f(z)$. Calculate $w = Df(z)(v)$ and show that

$$\langle v,v\rangle = g_{(x,y)}(w,w),$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product and $g_{(x,y)}$ is given at the start of the question.

(d) We equip $S^2\backslash\{(0,0,1)\}$ with the Riemannian metric induced by the Euclidean inner product in $\mathbb{R}^3$. Using the fact that $f : S^2\backslash\{(0,0,1)\} \to M$ is an isometry (you do not need to prove this), decide whether (M,g) is geodesically complete and justify your answer.