Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4181-WE01

Title: Mathematical Finance IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Consider a one-period financial market on two assets, where the risk-free asset price satisfies $B_0 = 1, B_1 = 4/3$ and the risky asset price satisfies $S_0 = 8$,

$$S_1 = \begin{cases} 12 & \text{with probability } 0.8 \\ 6 & \text{with probability } 0.2. \end{cases}$$

   (a) Is this market arbitrage free? Explain your answer.

   (b) Show that this market is complete and find the replicating portfolio for a general contingent claim that returns A if $S_1 = 12$ and B if $S_1 = 6$.

   (c) A "secured investment" derivative is offered on this market, with price $C > 0$ at time 0, which returns the initial investment C if $S_1 < S_0$, and returns amount $(1 + \alpha)C$ if $S_1 \geq S_0$. Find the value of α necessary to avoid arbitrage.

2. Let C be the current price of a European call option with strike price K and expiry date T, and let P be the current price of a European put option on the same stock with the same strike price and expiry date. Assume that interest is compounded continuously at rate r. Show that P and C satisfy

$$P + S_0 = C + K\mathrm{e}^{-rT},$$

   where S_0 is the current share price of the underlying stock.

3. Consider a financial market $\mathcal{M} = (B_t, S_t)$ on two assets. Suppose the current share price of the risky asset is $S_0 = 80$ and evolves according to the binomial model with $u = 1.25, d = 0.75, p_u = 3/4, p_d = 1/4$. Suppose that the current price of the risk-free asset is $B_0 = 1$ and the interest rate is $r = 0.2$.

   (a) Calculate the risk-neutral measure for this market.

   (b) Calculate the no-arbitrage prices at times $t = 0, 1, 2$ of a lookback call option on this market with payoff $S_T - S_{\min}$ at time $T = 3$.

4. (a) State the definition of a Brownian motion.

   (b) Show that if $(W_t)_{t\geq 0}$ is a Brownian motion, then $(\frac{1}{2}W_{4t})_{t\geq 0}$ is also a Brownian motion.

   (c) Show that if $(W_t)_{t\geq 0}$ is a Brownian motion, the process $W_t^3 - 3tW_t$ is a martingale with respect to the natural filtration $\{\mathcal{F}_t, t \geq 0\}$ of W_t.

5. (a) State Lévy's Theorem for recognising a Brownian motion.

   (b) Suppose $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are two independent Brownian motions and define

$$U_t = \theta X_t + \sqrt{1-\theta^2}\,Y_t$$
$$V_t = \sqrt{1-\theta^2}\,X_t - \theta Y_t$$

   where θ is a constant satisfying $0 < \theta < 1$. Show that $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are both Brownian motions.

   (c) Are $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ independent? Justify your answer using the box calculus.

6. Suppose $X \sim N(0, \sigma^2)$ is a Normal random variable under the measure $\mathbb{P}$. Let $\mathbb{Q}$ be an equivalent measure under which $Y = X + \theta\sigma^2$ is a Normal random variable with mean 0 and variance σ^2.

   (a) Find an expression for the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ as a function of X.

   (b) Calculate the expectation of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ under the measure $\mathbb{P}$.

   (c) Find $\mathbb{E}_{\mathbb{Q}}[e^{tY}]$.

SECTION B

7. The price of a risky asset over the time period $[0, T]$ is modelled by an n-period binomial model with parameters $u = \exp(\sigma\sqrt{T}/\sqrt{n})$, $d = u^{-1}$ and $p_u = p_d = 1/2$, and initial price S_0.

   (a) Describe the distribution of the price of the risky asset at time T.

   (b) Assuming an interest rate per step of rT/n, show that for large n the martingale probabilities for this model are approximately

$$q_u \approx \frac{1}{2} + \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma\sqrt{n}}, \quad q_d \approx \frac{1}{2} - \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma\sqrt{n}}.$$

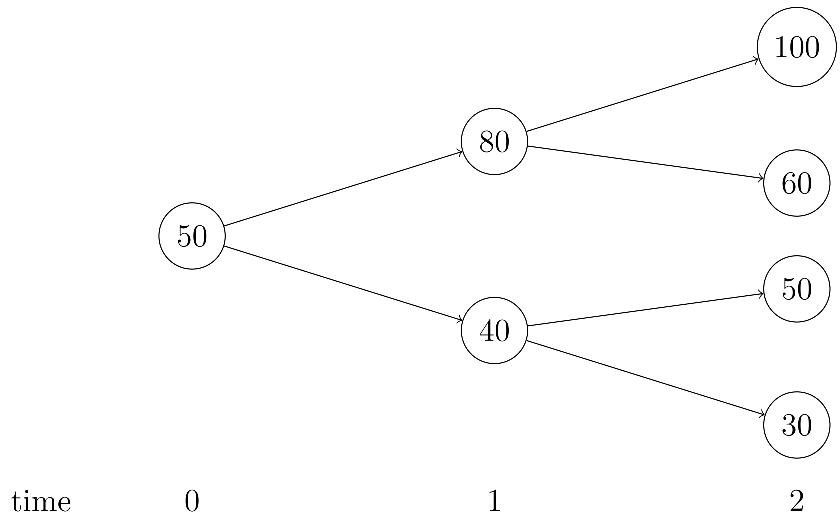
   (c) Using the risk-neutral valuation formula, or otherwise, deduce the Cox–Ross–Rubinstein formula for the price at time 0 of a European call option on this asset with strike price K and expiry time T.

   (d) What happens to the formula in part (c) as $n \to \infty$?
   [Hint: For $X \sim \mathrm{Bin}(n, p)$ with $0 < p < 1$,

$$\mathbb{P}[X \geq x] \to N\left(\lim_{n\to\infty} \frac{np - x}{\sqrt{np(1-p)}}\right) \quad \text{as } n \to \infty,$$

   where $N(\cdot)$ is the cumulative distribution function of a standard Normal random variable.]

8. Consider a 2-period financial market with possible share prices $S_t, t = 0, 1, 2$, of a risky asset given by the tree:



Suppose the interest rate per time step is $r = \frac{1}{5}$.

(a) What is the price at time 0 of a European put option with strike price 55 and expiry time 2?

(b) What is the price at time 0 of an American put option with the same strike price 55 and expiry time 2?

A Bermudan option gives the holder the ability to exercise the option only at pre-determined times $T_1, T_2, \ldots, T_n = T$ up to the expiry time T.

(c) What is the price at time 0 of a Bermudan put option with strike price 55 which can only be exercised at times 0 or 2? Comment on the relative sizes of the three prices you have calculated.

(d) Prove that a Bermudan call option has the same price as a European call option with the same strike price and expiry time.

9. Consider the Black-Scholes model

$$\begin{cases} dB_t = rB_t dt, \\ dS_t = \alpha S_t dt + \sigma S_t dW_t \end{cases}$$

in the time horizon $[0, T]$. Here r is the risk-free interest rate, α and σ are two constants, and $(W_t)_{t\geq 0}$ is a Brownian motion under the real world measure.

(a) Describe the dynamics of S_t under the risk-neutral measure.

(b) Find the solution of this SDE.

(c) Find the no-arbitrage price at time t of the contingent claim $\Phi(S_T) = S_T^3$.

(d) Calculate the hedging portfolio $(a_t, b_t)_{t\in[0,T]}$ that replicates the contingent claim in part (c).

10. (a) State Itô's Lemma for a smooth function $f(t, X_t)$ of time t and an Itô process $(X_t)_{t\geq 0}$.

(b) Prove that the following process

$$X_t = e^{-\mu t}X_0 + \theta\{1 - e^{-\mu t}\} + e^{-\mu t}\int_0^t \sigma e^{\mu s} dW_s$$

is a solution of the following SDE

$$dX_t = -\mu(X_t - \theta)dt + \sigma dW_t$$

where X_0 is a real constant, and θ, μ and σ are constant parameters, and $(W_t)_{t\geq 0}$ is a Brownian motion.
[Hint: apply Itô's Lemma to $e^{\mu t}X_t$]

(c) Calculate $\mathbb{E}[X_t]$ and $\mathbb{V}\mathrm{ar}[X_t]$.

SECTION C

11. In this question, let z_α be the critical z-value defined by $\mathbb{P}[Z > z_\alpha] = \alpha$, where Z is a standard Normal random variable.

(a) Describe carefully a Monte Carlo algorithm for producing an approximate $100(1-p)\%$ confidence interval of the no-arbitrage price at time 0 for a European call option with strike price K and expiry time T on an underlying risky asset. Give the confidence interval in terms of the usual unbiased estimators for mean and variance, which you should define, and z_α for an appropriate value of α. State clearly any assumptions you make about the behaviour of price of the underlying asset.

(b) Calculate the width of the approximate 95% confidence interval generated by your algorithm for the parameters $S_0 = 10$, $\sigma = 0.2$, $r = 0.1$, $K = 11$, $T = 1$, using the following sample of size 6 from a standard Normal distribution:

$$-1.728, 0.668, 0.291, 0.059, -0.267, 0.157$$

[You may find some of the following values of the standard normal cumulative distribution function useful.]

z	1.28	1.645	1.96
$N(z)$	0.9	0.95	0.975

(c) How many samples would you expect to need to use in order to halve the width of the confidence interval?