Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2019	MATH4221-WE01

Title:

Numerical Differential Equations IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2019

SECTION A

1. Consider the initial value problem $\dot{x} = f(x,t)\,,\ x(t_0) = x_0\,,\ t > t_0$.

   (a) Define the local truncation error for a multi-step numerical scheme.

   (b) Using finite difference, derive the Euler Backward method.

   (c) Let $|\ddot{x}|$ be bounded. Show that the order of the Euler Backward method is 1.

2. Consider a numerical method

$$x_{n+1} = x_n + k\left[(1-\theta)f(x_n,t_n) + \theta f(x_{n+1},t_{n+1})\right],$$

   where $0 < \theta < 1$, applied to an initial value problem $\dot{x} = f(x,t)\,,\ x(t_0) = x_0\,,\ t > t_0$.

   (a) Explain whether the above method is:

      i. explicit or implicit,

      ii. single-step or multi-step.

   (b) Give the general form of a method of the Runge-Kutta family.

   (c) Show that the above method can be written in the form of a Runge-Kutta method.

   (d) Write down the Butcher tableau corresponding to this method.

3. Consider the numerical scheme

$$x_{n+2} = 3x_{n+1} - 2x_n - kf_n$$

   applied to the initial value problem $\dot{x} = f(x,t)\,,\ x(t_0) = x_0\,,\ t > t_0$.

   (a) Write down the characteristic polynomials.

   (b) Give the definition of a consistent numerical scheme and show that the above scheme is consistent.

   (c) Give the definition of a zero-stable numerical scheme and determine whether the above scheme is zero-stable.

4. Let $u = u(x)$ solve the following boundary-value problem

$$u''(x) + (x+1)u(x) = x^2 + 1 \quad \text{for } x \in [0,1],$$
$$u'(0) - 2u(0) = 1, \qquad u'(1) = 0.$$

(a) Assuming that $u \in C^3([0,1])$, write down the Taylor expansion for $u(h)$ centred at 0 with accuracy $O(h^3)$, taking into account that u satisfies the differential equation.

(b) Based on (a), find a second-order approximation for the boundary condition $u'(0) - 2u(0) = 1$ in terms of $u(0)$ and $u(h)$. Justify your approximation.

5. For the equation $\partial_t u(x,t) + a\partial_x u(x,t) = 0$ (a is a real constant), consider the following approximation scheme

$$\frac{u_m^{n+1} - u_m^n}{\tau} + a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} = 0 \quad \text{with } n \in \mathbb{Z}_+,\ m \in \mathbb{Z}. \tag{1}$$

(a) Considering a particular solution $u_m^n = \lambda(\phi)^n e^{im\phi}$, where $\phi \in [0, 2\pi]$, to the problem (1), find the amplification factor $\lambda(\phi)$.

(b) Assuming that $\tau = rh/a$ for some constant r and using the spectral stability test, determine under which conditions on the parameter r the scheme (1) is stable.

6. (a) Consider the iteration

$$x^{k+1} = Gx^k + c$$

where G is a square matrix, and x^j and c are vectors, with x^0 given. Assuming that the equation $x = Gx + c$ has a unique solution, state necessary and sufficient conditions on G that guarantee the convergence of the iteration.

(b) Given a real matrix

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{pmatrix} \quad \text{with } \alpha \neq 0, \tag{2}$$

and some vector $b \in \mathbb{R}^3$, we seek to solve $Ax = b$ using the Jacobi iteration process. Find the transition matrix G corresponding to A.

(c) Find all values of parameters α and β such that the Jacobi iteration process converges for arbitrary initial vector x^0.

SECTION B

7. Consider the initial value problem

$$\ddot{x} + \frac{3}{2}\dot{x} = x\,,\ x(0) = 0\,,\ \dot{x}(0) = 1\,,\ t > 0.$$

(a) By introducing new variables $x_1 = x$ and $x_2 = \dot{x}$ write the system in the form of $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\,,$$

and A is a square matrix and hence find the analytical solution of the initial value problem.

(b) Find an explicit formula for the approximate solution $\mathbf{x}_n$, produced by the Euler Forward method with step-size $k > 0$.

(c) Show that the solution derived by the Euler Forward method converges to the analytical solution for $k \to 0$, in a fixed finite interval $t \in [0, T]$.

8. Consider an initial value problem $\dot{x}(t) = f(x(t), t)\,,\quad x(0) = x_0\,,\quad t > 0$. Let p_s be a polynomial of degree s interpolating $f(x(t), t)$ at $s + 1$ nodes $t_i = ik$, $(i = 0, \ldots, s)$:

$$p_s(t_i) = f(x(t_i), t_i)\,,\quad i \in \{0, \ldots, s\}\,.$$

This polynomial can be written in the Lagrange form

$$p_s(t) = \sum_{i=0}^{s} f(x(t_i), t_i)\ell_i(t)\,,\ \text{where}\ \ \ell_i(t) = \prod_{\substack{j=0\\ j\neq i}}^{s} \frac{t - t_j}{t_i - t_j}\,. \tag{3}$$

(a) Use equation (3) along with a suitable change of variables to show that the s-step Adams-Bashforth method is

$$x_{n+1} = x_n + k\sum_{i=0}^{s-1} b_i f_{n-i}$$

where

$$b_i = \int_0^1 \prod_{\substack{j=0\\ j\neq i}}^{s-1} \frac{\tau + j}{j - i}\, d\tau\,.$$

(b) Derive the two-step Adams-Bashforth method.

(c) Show that the s-step Adams-Bashforth method has order s.

9. Let $u = u(x) \in C^1([0,1])$ be a real-valued function.

   (a) Assuming that $u(0) = u(1) = 0$, state the continuous version of the Poincare inequality for u.

   (b) Prove that
   $$\sup_{x\in[0,1]} |u(x)| \le \int_0^1 |u'(y)|\,dy + \left|\int_0^1 u(y)\,dy\right|.$$
   The assumption $u(0) = u(1) = 0$ is not needed here.
   Hint: consider $|u(x) - \int_0^1 u(y)\,dy|$.

   Let u_i, $i \in \{0, \ldots, N\}$ be a real valued grid function.

   (c) Assuming that $u_0 = u_N = 0$, state the discrete version of the Poincare inequality for u_i, $i \in \{0, \ldots, N\}$.

   (d) Prove a discrete analogue of (b), namely
   $$\max_{i\in\{0,\ \ldots,\ N\}} |u_i| \le \left|\sum_{k=1}^{N-1} hu_k\right| + \sum_{k=1}^{N} |u_k - u_{k-1}|,$$
   where $hN = 1$. The assumption $u_0 = u_N = 0$ is not needed here.

10. Let $N \in \mathbb{N}$, $N \ge 2$. Consider the following eigenvalue problem for a difference scheme
$$\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} = -\lambda u_k \quad \text{for } 1 \le k \le N-1 \quad \text{with } h(N-1) = 1, \tag{4}$$
$$u_0 = -u_1, \quad u_N = -u_{N-1}. \tag{5}$$

   (a) Find the roots μ_1 and μ_2 of the characteristic polynomial associated to the difference equation (4). Find the values of $\mu_1 + \mu_2$ and $\mu_1\mu_2$.

   (b) Under the assumption $\mu_1 = \mu_2 := \mu$, the general solution to the difference equation (4) takes the form
   $$u_k = c_1\mu^k + c_2 k\mu^k.$$
   Taking into account the boundary conditions (5), find all λ such that the problem (4)–(5) has a nonzero solution.

   (c) Under the assumption $\mu_1 \ne \mu_2$, the general solution to the difference scheme takes the form
   $$u_k = c_1\mu_1^k + c_2\mu_2^k.$$
   Taking into account the boundary conditions (5), find all λ such that problem (4)–(5) has a nonzero solution.

SECTION C

11. Consider a Linear Multistep Method

$$\sum_{j=0}^{s} \alpha_j x_{n+j} = k \sum_{j=0}^{s} \beta_j f_{n+j}\,,$$

applied to the ordinary differential equation $\dot{x} = \lambda x$. We define the characteristic polynomials

$$\rho(\zeta) = \sum_{j=0}^{s} \alpha_j \zeta^j$$

and

$$\sigma(\zeta) = \sum_{j=0}^{s} \beta_j \zeta^j\,.$$

(a) Give the definition of the *region of absolute stability* for the above Linear Multistep method.

(b) Consider the leapfrog method

$$x_{n+1} = x_{n-1} + 2kf(x_n, t_n)\,.$$

i. Write down the values of s, α_j, β_j.

ii. Find the region of absolute stability corresponding to this method.

iii. Sketch the region of absolute stability on the complex plane.