Durham University

EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May/June	2020	MATH1051-WE01

Title:
Analysis I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 Let $a, b \in \mathbb{R}$ and $c > 0$. Show that

1.1 $|a \cdot b| = |a| \cdot |b|$.

1.2 $|a| > c$ is equivalent to $(a > c$ or $a < -c)$.

1.3 $|3a - 5| > a + 9$ is equivalent to $|a - 3| > 4$.

Q2 Calculate the limits of the following sequences, stating any results from the lectures that you use.

2.1 $a_n = \frac{\sqrt{4n^2-3}}{n+1}$

2.2 $b_n = \frac{n-1}{(n+2)^2-(n-2)^2}$

2.3 $c_n = \frac{1+2+\cdots+(n-1)+n}{n^2}$

Q3 **3.1** Let

$$X = \left\{ \frac{n^2 - 4n + 3}{n^2 + 1} \in \mathbb{R} \;\middle|\; n \in \mathbb{N} \right\}.$$

Calculate $\sup(X)$ and $\inf(X)$. Justify your statements.

3.2 Let $X \subset \mathbb{R}$ be a set, and $C \in \mathbb{R}$ such that

(i) For all $x \in X$ we have $x \leq C$, and

(ii) For every $B < C$ there exists an $x \in X$ with $B < x$.

Show that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X with

$$\lim_{n \to \infty} x_n = C.$$

Q4 **4.1** State the Bolzano–Weierstrass Theorem.

4.2 Give an example of an unbounded sequence $(x_n)_{n \in \mathbb{N}}$ which has a subsequence converging to 0, and a subsequence converging to 1. Justify your statement.

Q5 Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function.

5.1 Assume that $f(a) > 5$ for some $a \in \mathbb{R}$. Show that there exists a $\delta > 0$ such that $f(x) > 5$ for all $x \in \mathbb{R}$ with $|x - a| < \delta$. Hint: Consider $\varepsilon = f(a) - 5$.

5.2 Define $g \colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \leq 5 \\ 5 & \text{if } f(x) > 5 \end{cases}$$

Show that g is a continuous function. Hint: For $a \in \mathbb{R}$, distinguish the cases $f(a) > 5$, $f(a) < 5$, and $f(a) = 5$.

Q6 **6.1** Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ be fixed. Assume we have

$$f(x) = f(c) + (x-c)^2 f_1(x)$$

with a differentiable function $f_1 : \mathbb{R} \to \mathbb{R}$. Show that f is twice differentiable at $x = c$ and express $f''(c)$ in terms of f_1.

6.2 Calculate

$$\lim_{x \to \frac{\pi}{2}} \frac{\frac{e}{\pi}x^2 - \frac{e\pi}{4} + \int_x^{\pi/2} e^{\sin(t)}dt}{1+\cos(2x)}.$$

Q7 **7.1** Determine for each of the following series whether they are convergent and whether they are absolutely convergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \log(n)}{n} \qquad \text{and} \qquad \sum_{n=3}^{\infty} \sqrt{\frac{n-3}{n^5+1}}.$$

7.2 Compute the radius of convergence of the following power series:

$$f(x) = \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

and give a power series for the functions $g(x) = f'(x)$ and $h(x) = g(2x)$.

Q8 Let $I = [a, b] \subset \mathbb{R}$ be a closed and bounded interval.

8.1 Give the definition of a uniformly continuous function on I.

8.2 Give the definition of a regulated function $f : I \to \mathbb{R}$ and explain how to define its integral $\int_a^b f(x)dx$.

8.3 Show that every uniformly continuous function on I is regulated.

Q9 **9.1** Show that

$$\lim_{k \to \infty} \int_0^{2\pi} \frac{k \sin(kx)}{x^2 + k^2} dx = 0.$$

9.2 Find all real values $c \in \mathbb{R}$ for which the improper integral

$$\int_0^{\infty} \frac{x^c}{\sqrt{x^2 + x}} dx$$

converges.

Q10 Let $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n - x^{n+1}$.

10.1 Calculate the pointwise limit f of the functions f_n.

10.2 Determine the global maximum of the function f_n.

10.3 Decide whether the sequence f_n converges uniformly to f on $[0, 1]$. Justify your answer.