Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2020	MATH1061-WE01

Title: Calculus I

Time (for guidance only):	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: There is no restriction on the model of calculator which may be used.

Instructions to Candidates:	Credit will be given for your answers to all questions. All questions carry the same marks. Please start each question on a new page. Please write your CIS username at the top of each page. Show your working and explain your reasoning.

Revision:	

ED01/2020

Q1 **1.1** Without using Taylor series or L'Hôpital's rule, calculate the limit

$$\lim_{x\to 6} \frac{x-6}{\sqrt{8x+1}-7}.$$

1.2 Use Taylor series to calculate the limit

$$\lim_{x\to 0} \frac{2\sin(3x)-6x\cos(x^2)}{\sin(2x^3)-x\log(1+x^2)}.$$

1.3 Give an example of a function $f(x)$ that satisfies all of the following conditions:
$f(x)$ is continuous at $x=1$,
$f(x)$ is not differentiable at $x=1$,
$f(x)$ is not continuous at $x=2$.
You don't need to prove that your function satisfies these conditions.

Q2 **2.1** Prove that $\left|\cos x - 1 + \frac{1}{2}x^2\right| \leq \frac{2}{3}$ for all $x \in [0,2]$.

2.2 Solve the initial value problem,

$$y' + \frac{2}{3x}y + x^4y^4 = 0, \qquad y(1) = \frac{1}{2}.$$

Q3 For $x > 0$, find the general solution $y(x)$ of the ordinary differential equation

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}.$$

Q4 **4.1** Calculate the integral

$$\int_{-2}^{2} \frac{x^4}{1+e^{\sinh x}}\,dx.$$

4.2 Let D be the triangle in the (x,y) plane with vertices $(0,0)$, $(-1,2)$, $(1,2)$. Calculate the double integral

$$\iint_D e^{y^2}\,dxdy.$$

Q5 The function $f(x)$ has period 2, that is $f(x+2)=f(x)$, and is given by $f(x)=1+x$ for $-1<x<1$.

5.1 Calculate the Fourier series of $f(x)$.

5.2 By evaluating the Fourier series at $x=\frac{1}{2}$, calculate

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1}.$$

Q6 Let $f(x, y)$ be a differentiable function on $\mathbb{R}^2$. A change of coördinates $(x, y) \to (u, v)$ is given by

$$x = \sqrt{uv}\ ,\ y = \sqrt{\frac{u}{v}},$$

where x, y, u and v are all positive. By expressing u and v as functions of x and y, use the chain rule to find expressions for f_x and f_y in terms of f_u, f_v, u, and v. Show that

$$x^2\left(\frac{\partial f}{\partial x}\right)^2 + y^2\left(\frac{\partial f}{\partial y}\right)^2 = 2u^2\left(\frac{\partial f}{\partial u}\right)^2 + 2v^2\left(\frac{\partial f}{\partial v}\right)^2.$$

Q7 Find and classify the stationary points of the function

$$f(x, y) = xy^2 - 4xy + 3x + x^2.$$

Q8 Consider the differential equation.

$$(4 - x^2)\frac{d^2y}{dx^2} + \lambda y = 0.$$

8.1 Explain what is meant by a *regular point* of a differential equation. Which points are regular?

8.2 A series solution for the equation takes the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Derive a recurrence relation for the coefficients a_n.

8.3 Show that the equation admits a solution $P_m(x)$ which is a polynomial of order m in the case that $\lambda = m(m-1)$, and find explicit expressions for $P_2(x)$ and $P_3(x)$.

Q9 A one-dimensional bar of length π lies along the x-axis between $x = 0$ and $x = \pi$. The temperature in the bar $T(x, t)$ obeys the heat equation

$$\frac{\partial T}{\partial t} = k^2\frac{\partial^2 T}{\partial x^2}.$$

The ends of the bar are insulated so that $T_x(0, t) = T_x(\pi, t) = 0$.

9.1 Show that

$$T(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)e^{-n^2k^2t}$$

satisfies the heat equation and the boundary conditions at $x = 0$ and $x = \pi$.

9.2 If at $t = 0$, the temperature in the bar is given by

$$T(x, 0) = \begin{cases} 0 & \text{for } 0 \le x \le \pi/2 \\ 1 & \text{for } \pi/2 < x \le \pi \end{cases}$$

find the temperature $T(x, t)$ by calculating a_n in the above Fourier expansion, and write down explicitly the first three non-zero terms in the expansion.

Q10 The function $\theta(x,t)$ satisfies

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} - \theta.$$

The Fourier transform of θ is defined by

$$\bar{\theta}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}\theta(x,t)dx.$$

10.1 Use the differential equation to obtain $\bar{\theta}(k,t)$ in terms of its initial value $\bar{\theta}(k,0)$.

10.2 If $\bar{\theta}(k,0) = e^{-k^2}$ find $\theta(x,t)$.

(You may use the result

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha k^2 + ikx}dk = \frac{1}{2\sqrt{\pi\alpha}}e^{-x^2/(4\alpha)}.)$$